Stochastic pathwise approach to imitational behavior [1]

Yasuo Tanabe
Faculty of Economics, Nagasaki University,
4-2-1, Katafuchi, Nagasaki, 850-8506, Japan

July, 2002

[1] This paper benefits from comments on my previous paper (Tanabe (2001)) by Hiroshi Oaku at the Japanese Economic Association held on 7-8 October 2001.

Abstract

We study the imitational behavior, that is, learning or imitation of each individual in a multi-population game when the population state converges to an equilibrium state. Our study is based on the pathwise analysis of a continuous time Markov chain that completely describes the imitational behavior of each individual.

We show that all individuals settle on a pure strategy in the long run if and only if the pure strategy is a best reply to the limit state. Moreover, all individuals' time average of holding each strategy become equalized to the weight of the strategy at the limit state to which the population state converges, because their schemes of imitational behavior go to the same. Our results assert that it is inappropriate to reason about each individual's imitational behavior in the population based only on the convergence of the population state.

JEL classification: C72,C79.

Keywords: imitational behavior,propagation of chaos, Mckean process,elimination of strategy,ergodicity, equalization.

1 Introduction

In recent years many authers study imitation or learning models in which individuals choose actions by imitating or learning others'. For example, Börgers and Sarin (1997) and (2000), Cabrales (2000) and Gale et al.(1995) present models in which individuals randomly select another individual and imitate his strategy when the satisfaction level with the payoffs from their current strategies fall below some target level.

Björnerstedt and Schlag (1996) and Schlag (1998) present models in which the payoff of individuals are realized by a multi-armed bandit and they choose an action (to pull an arm of the bandit) based upon the imformations about payoffs of their own and another individual ramdomly sampled.

Weibull (1995) and Bjönerstedt and Weibull (1996) formulate social evolution by imitation in a generic scheme and give a few specific examples of imitation dyanamics. In these works it is shown that imitation or learning models can be reduced to the replicator dynamics in certain settings.

In the works they model imitational behavior,i.e.,imitation or learning in stochastic formulations, where individuals choose their strategies on the basis of some probabilistic law, and deduce deterministic equations which describe dynamics of population share of individuals to use each strategy. Then they get some implications about the imitational behavior of inidividuals by the analysis of the deterministic dynamics. This approach provides sufficient informations about aggregated behavior in the populations. But it does insufficient informations about the stochastic behavior of each individual such as whether the strategy of each individual converges to some pure strategy or whether each individual goes to behave alike by imitational behavior. This motivates us to analyze a stochastic process itself that represents the imitational behavior of each individual.

In this article we construct and analyze the stochastic process that perfectly describes the imitational behavior of each individual [1]. This is done in a generic frame basically borrowed from Weibull (1995) and Björnersted and Weibull (1996).

[1] In this article we extend the stochastic pathwise approach of Tanabe (2001) in two aspects. One is the extention from a single-population to a multi-population in the model. The other is that results are explicitly applied to not only replicator dynamics but payoff monotonic dynamics which contain the replicator dynamics as a special case.

In the frame we consider n populations with countable individuals who are infinitely lived. Each individual holds some pure strategy for some time interval, and occasionally reviews and changes his strategy. This reset of the strategy is based on some review rate function and choice probability function which may depend on the payoff of his strategy against the current population state in n-player game.

The stochastic process constructed for our analysis is a continuous time Markov chain on the space of pure strategies. It jumps from one pure strategy to another, following the choice probability when an arrival time of a Poisson process comes, of which arrival rate is given by the review rate function. Each stochastic path of the process represents the realized transition of each individual's strategy, and the marginal distribution of the process at each time does the population shares. Therefore, by analyzing the process in a stochastic pathweise we can get implications about the realized imitational behavior of each individual.

To put it concretely, consider the following two questions in this article. In the study of imitation or learning model, one of the most interesting issue is what the outcome by imitation or learning is, that is, as the result of imitation or learning process, to what equilibrium state the population state converges in the long run. These questions are to investigate the behavior of each individual in the convergence of the population state to such an equilibrium state.

The first question is on the converegence of the strategy taken by each individual to some pure strategy. When the population share of individuals who take pure strategy k converges to one in the long run, does the strategy of each individual converge to strategy k, that is, does each individual go to stick on strategy k ?

The second is on the time-averaged behavior of each individual. When the state of population shares converges to some state which does not necessarily put all weight on one pure strategy, what is each individual's time average of holding strategies, or in what ratio of time does each individual hold each strategy in other words ?

In order to answer these questions, the analysis of deterministic dynamics which represents the whole population behaviors are insufficient, but we need to analyze the stochastic process in a pathwise.

Our main results are as follows. The first one is that under some assumptions, the strategy of each individual converges to some pure strategy if and only if the strategy is a unique best reply to the limit state to which the population state converges. In a single population model this means that even if the population share goes on strategy k, the strategy of each individual does *not* converge to strategy k unless (k, k) is a strict Nash equilibrium. So the answer to the first question is "no" in a mathematical sence. This result asserts that it is inappropriate to reason about each individual's imitational behavior in the population based only on the convergence of the population state.

The second is that each individual's time average of holding time of a strategy converges to the weight of the strategy at the limit state when the population share of individuals converges to some state. This result is an extension of Birkoff's individual ergodic theorem in mathematics and tells us two things. One is that the behavior of all individuals become equalized at least in the time average when the population share goes to some state. The other is that the time average is given by the limit state.

So the second result implies each individual goes to behave alike at least in the time average by imitational behavior in the situation where the population share converges. Consequently the time average of payoff for each individual equals to that of the limit state. Concering about the first question, even if the strategy of each individual does not converges to strategy k, the mean visiting time to strategy k equals to one, and each individual stays on the strategy in all time except occasional visits to other strategies if the population share goes on strategy k.

The rest of this paper is organized as follows. Section 2 describes the model and makes preparations for the anlysis. Section 3 constructs the stochastic process called a Mckean process that describes the imitational behavior of each individual. Section 4 and 5 present the results about the elimination of a strategy and the ergodicity of the process, containing our two main results. Section 6 demonstrates how our results work in applications. Section 7 concludes, and Appendix contains all proofs.

2 Formulation

In this section we model imitational behavior in a stochatic frame basically borrowed from Weibul(1995) and Björnersted and Weibull (1995). Further, we set some technical assumptions and specific types of imitational schemes to realize regular selection dynamics.

2.1 Model

Here we give basic notations and briefly sketch the model. First we set a n- player game as follows. $I = \{1, \dots, n\}$ is the set of players, and $S^i = \{1, \dots, m^i\}$ with $m^i \geq 2$ be the pure-strategy set of player $i \in I$ using character i, j for a player and h, k, l for a pure strategy. $\triangle^i$ is the mixed-strategy set on S^i, and pure strategy $h \in S^i$ is idetified with the unit vector $e^i_h = (0, \dots, 0, \underset{h-th}{1}, 0, \dots, 0)$. We denote the set of pure strategy profile $\times^i S^i$ by S and the polyhedron of mixed-strategy profiles by Θ with open domain $D \subset R^m$ containing Θ , where $m = m^1 + \cdots + m^n$. $\pi^i(u)$ is the payoff to player $i \in I$ when $u = (u^1, \dots, u^n) \in \Theta$ is played.

Now we introduce a process of imitational behavior. In the preliminary, first suppose n populations with Λ individuals suffixed by $\lambda = 1, \dots, \Lambda$ who live forever and interact each other. We identify $u \in \Theta$ a population state profile, i.e., u^i_h is the share of individuals on population i who uses strategy $h \in S^i$. Each individual in each population holds some pure strategy for some time interval, and occasionally reviews and changes his strategy based on review rate functions $r^i_h : D \to R_+$ and choice probability functions $p^i_h : D \to \triangle^i, i \in I$. We call (r^i_h, p^i_h) *imitation scheme for h-individual in population i* and $(r^i, p^i), i \in I$ simply *imitation scheme*.

Let $X^i_\lambda(t)$ be the strategy of λ-th individual in population i at time t and define an empirical distribution on S^i and S by $U^{i(\Lambda)}(t) = \dfrac{1}{\Lambda}\sum_{\lambda=1}^{\Lambda} \delta_{X^i_\lambda(t)}$ and $U^{(\Lambda)}(t) = (U^{1(\Lambda)}(t), \dots, U^{n(\Lambda)}(t))$ respectively, where δ_x stands for δ-measure at x. Then individuals change their strategies one by one in the following way.

Let $N^{i,1}_\lambda(t), \lambda = 1, \dots, \Lambda, i \in I$ be mutually independent Poisson processes independent of $X^i_\lambda(0), \lambda = 1, \dots, \Lambda, i \in I$ with intensity or arrival rate $r^i_{X^i_\lambda(0)}(U^{(\Lambda)}(0))$, $\lambda = 1, \dots, \Lambda, i \in I$ respectively, and suppose σ_1,the first jump time of $N^{i_1,1}_{\lambda_1}(t)$, occurs before that of $N^{i,1}_\lambda(t), \lambda \neq \lambda_1, i \neq i_1$. i.e., σ_1 is "the first of the first jump times”. Then at σ_1, only the λ_1-th individual in population i_1 reviews his strategy and chooses strategy $X^{i_1}_{\lambda_1}(\sigma_1)$ according to probability $p_{X^{i_1}_{\lambda_1}}(U^{(\Lambda)}(0))$.

Next, let $N^{i,2}_{\lambda}(t), \lambda = 1, \dots, \Lambda, i \in I$ be mutually independent Poisson processes independent of $\{X^i_\lambda(s), s \leq \sigma_1, \lambda = 1, \dots, \Lambda, i \in I\}$ with intensity $r^i_{X^i_\lambda(\sigma_1)}(U^{(\Lambda)}(\sigma_1))$, $\lambda = 1, \dots, \Lambda, i \in I$ respectively, beginning at σ_1, and suppose σ_2, “the first of the first jump times” is realized by $N^{i_2,2}_{\lambda_2}(t)$. Then at σ_2, only the λ_2-th individual in population i_2 reviews his strategy and chooses strategy $X^{i_2}_{\lambda_2}(\sigma_2)$ according to probability $p_{X^{i_2}_{\lambda_2}}(U^{(\Lambda)}(\sigma_1))$, and so on.

This reset procedure is the same for all individuals in each population. If $X^i_\lambda(0), \lambda = 1, \dots, \Lambda$ are independently and identically distributed for $i \in I$, by the law of large numbers, $U^{(\Lambda)}(t)$ converges to some $u(t) = (u^1(t), \dots, u^n(t)) \in \Theta$ in probability as $N \to \infty$. Moreover, for $i \in I$ $X^i_\lambda(t), \lambda = 1, \dots, \Lambda$, the transition of individuals’ strategy, go to represent a common stochastic process X^i that is constructed by review rates $r^i(u(t))$ and choice probabilities $p^i(u(t)), i \in I$ and has $u(t)$ as a marginal distribution. This phenomenon is called a *propagation of chaos*, and the common stochastic process is called a *Mckean process*. The propagation of chaos is equivarent to the law of large numbers for $U^{(\Lambda)}$ (see Tanaka (1983) and Snitzman(1984)).

Now we reset n populations with countable individuals who review and change thier strategies based on given imitation scheme $(r^i, p^i), i \in I$. The above argument suggests us to consider that the transition of all individuals is represented by $\bigotimes_{i=1}^{n} \overset{\infty}{\otimes} X^i$, i.e., an infinite direct product of independent copies of the Mckean process. Then, if we show some property of a stochastic path holds with probaility one for X^i, the realized transition of the strategy of each individual in poplation i always represents that property. Thus, by analyzing the Mckean process in a pathweise, we can understand the behavior of each inidividual in populations which is stochastically realized based on review rates $r^i(u(t))$ and choice probabilities $p^i(u(t)), i \in I$.

We construct the Mckean process and prove the propagation of chaos in section 3. After that we study the imitational behavior in an individual-wise based on the analysis of the Mckean process.

2.2 Assumptions

We assume that for $h \in S^i, i \in I$ the review rate function $r^i_h : D \to R_+$ is Lipschitz continuous with open domain $D \subset R^m$ containing Θ, and that
there exist $C^i_{r1}, C^i_{r2} > 0$ for $i \in I$ such that

$$C^i_{r1} \geq r^i_h(u) \geq C^i_{r2}, h \in S^i, u \in \Theta. \tag{2.1}$$

We remark that we always have the left inequality of (2.1) by the continuity of r^i on Θ. So assumption (2.1) essentially assures that each individual in population i reviews his strategy with positive probability even when the current state is on the boundary of Θ such as $u^i_h = 1$ for some $h \in S^i$. This is an implicit assumption that each individual seeks an opportunity to promote his payoff at any state, so that we consider it a rational assumption [2]. In the following we always assume (2.1) holds for all $i \in I$.

Further, for $h \in S^i, i \in I$ the choice probability function $p^i_h : D \to \triangle^i$ is also Lipschitz continuous with open domain D, and impose the following assumptions. These assumptions are not strong, so that many models of imitational behavior satisfy them.

(B1:i) There exists $C^i_{p1} > 0$ such that

$$C^i_{p1} u^i_h \geq p^i_{lh}(u), h \neq l \in S^i, u \in \Theta.$$

(B2:i) There exists $C^i_{p2} > 0$ such that

$$p^i_{lh}(u) \geq C^i_{p2} u^i_h, h \neq l \in S^i, u \in \Theta.$$

(B3:i) There exists Lipschitz continuous function $p^i : D \to \triangle^i$ such that

$$p^i_{lh}(u) = p^i_h(u), h, l \in S^i, u \in \Theta.$$

Assumption (B1:i) and (B2:i) are technical ones for the analysis. Roughly speaking, put together, they imply $p^i_h(u)$ is in propotion to u^i_h. Assumption (B3:i) prescribes that the choice probability is the same for all individuals in population i. As is seen in a later counterexample, this assumption implicitly excludes the

[2] But an alternative assumtion that the review rate degenerates into zero at the state of $u^i_h = 1$ is also attractive.

limit state at which the corresponding stochastic process X is a reducible Markov chain. This assumption makes the analysis much easier.

2.3 Imitation schemes for regular selection dynamics

In section 3 it will be shown that the imitation dynamics given Weibull(1995) and Björnersted and Weibull (1996) is deduced as the marginal distribution of the Mckean process for the imitational behavior. On the other hand, regular selection dynamics have been studied as one of important classes of dynamics for social evolutions. In this subsection we set two classes of imitaion schemes for the marginal distribution of the associated Mckean processes to represent regular selection dynamics. Further we give a few specific examples of these classes.

2.3.1 Two classes of imitaion schemes

The regular selection dynamics are presented for application to social contexts and more flexible than the replicator dynamics which are formulated originally in a biological nature.

The regular selection dynamics on Θ is generally given by [3]

$$\dot{u}^i_h = G^i_h(u)u^i_h, h \in S^i, i \in I, \tag{2.2}$$

where $G^i_h, h \in S^i, i \in I$ are Lipshcitz continuous functions with open domain D containing Θ and satisfy

$$\sum_{l \in S^i} G^i_l(u)u^i_l = 0, u \in \Theta, i \in I. \tag{2.3}$$

R^{m^i}-valued function G^i is called *regular growth rate function*, and the condition (2.3) is to ensure the solution orbit remains in Θ.

A R^{m^i}-valued function F on Θ is called *payoff monotonic* (weibull (1995)) or simply *monotonic* (Samuelson and Zhang (1992)) if for any $h, h' \in S^i, u \in \Theta, i \in I$,

$$\pi^i(e^i_h, u^{-i}) \geq \pi^i(e^i_{h'}, u^{-i}) \iff F_h(u) \geq F_{h'}(u).$$

[3]See Samuelson and Zhang (1992) and subsection 5.5 in Weibull (1995).

When regular growth rate function G^i is payoff monotonic, the associated selection dynamics (2.2) is called *payoff monotonic*. We focus on two classes of imitation schemes which are not necessarily disjoint. The both classes are sufficiently broad so that the both of them realize all regular selection dynamics. That is, the following imitation dynamics given by Weibull(1995) and Björnersted and Weibull (1995) represent all regular selection dynamics [4] when they are based on imitation schemes in the both classes.

$$\dot{u}_h^i = \sum_{l \in S^i} r_l^i(u) p_{lh}^i(u) u_l^i - r_h^i(u) u_h^i, h \in S^i, i \in I. \tag{2.4}$$

Class I. One is that $r_h^i(u) = r, h \in S^i$ for some positive constant r and $p_{lh}^i(u) = \alpha_{lh}^i(u) u_h^i, h \neq l \in S^i$ for some $R_+^{m^i}$-valued, Lipschitz continuous function α_{lh}^i with $\sum_{l \neq h \in S^i} \alpha_{hl}^i(u) u_l^i \leq 1, u \in \Theta$. That is, the review rate is a common constant among individuals in population i, and the probabibility of changing strategy is propotional to population shares. For this case, (2.4) is reduced to

$$\dot{u}_h^i = r \sum_{l \neq h \in S^i} (\alpha_{lh}^i(u) - \alpha_{hl}^i(u)) u_l^i u_h^i, h \in S^i, i \in I. \tag{2.5}$$

Class II. The other is of a type such that only a choice probability fixed by $p_{lh}^i(u) = u_h^i, h, l \in S^i, i \in I$. That is, the selection of the strategy is random according to the frequency of the current population state. For the case, (2.4) is reduced to

$$\dot{u}_h^i = (\sum_{l \in S^i} r_l^i(u) u_l^i - r_h^i) u_h, h \in S^i, i \in I. \tag{2.6}$$

---

[4] In fact, if we set

$$r_h^i(u) = r, p_{lh}^i(u) = (G_h^i(u) + r) u_h^i / r, h \in S^i, i \in I$$

for some $r > \max_{h,i} \sup_{u \in \Theta} \{-G_h^i(u)\}$, (2.5) turns into (2.2). Further, setting

$$r_h^i(u) = \alpha - G_h^i(u), p_{lh}^i(u) = u_h^i$$

for some $\alpha > \max_{h,i} \sup_{u \in \Theta} G_h^i(u)$, transforms (2.6) into (2.2).

Then the growth rate functions are payoff monotonic if and only if so are $\{-r^i, i \in I\}$.

2.3.2 Examples

Formally any regular selection dynamics can be obtained as the marginal distribution of a Mckean process by two simple forms such that $r^i_h(u) = r$ or $p^i_{lh}(u) = u^i_h$. Here we present specific examples of imitational behavior model in the both forms.

Propotional imitation. This is based on a "multi-armed bandit" approach (Schlag(1998), Björnerstedt and Schlag(1996)). Given increasing Lipschitz continuous function f on R, for $h \in S^i, i \in I$, , let $P^i_h(u)$ be a continuous function from Θ into the space [5] of all probability distributions on R_+ such that $P^i_h(u)$ is supported with interval $[\underline{\omega}, \overline{\omega}](0 \leq \underline{\omega} < \overline{\omega})$ and has mean $P^i_h(u)$ is $f(\pi^i(h, u^{-i}))$.

When the current state is $u(t)$ at time t , the payoff for each h-individual on population i is independently drawn from $P^i_h(u(t))$ across individuals and time (by multi-armed bandit). When a reviewing time comes to an h-individual at time t , he randomly samples another individual in population i according to the probability $u^i(t)$.

Now suppose his payoff is x^i_h at the current state and he samples an h'-individual whose payoff is $x^i_{h'}$. Then he switches his strategy from h to h' with probability $\dfrac{x^i_{h'} - x^i_h}{\overline{\omega} - \underline{\omega}}$ only if $x^i_{h'} > x^i_h$, where $\dfrac{1}{\overline{\omega} - \underline{\omega}}$ is called a *switching rate*.

Setting $\alpha^i_{hh'}(u) = \dfrac{1}{\overline{\omega} - \underline{\omega}} \int_{x^i_{h'} > x^i_h} (x^i_{h'} - x^i_h) P^i_h(u)(dx^i_h) P^i_{h'}(u)(dx^i_{h'}), h' \neq h \in S^i$ for $u \in \Theta$, it holds that $p^i_{hh'}(u) = \alpha^i_{hh'}(u) u^i_{hh'}$ with $0 < \alpha^i_{hh'}(u) < 0$ for $h' \neq h \in S^i$ in this model.

Further, if we assume the review rate $r^i_h(u) = r, h \in S^i, i \in I$, the imitation scheme for this model is of class I and satisfies assumption (B1:i) and (B2:i) for

[5]The space is topologized with the total variation norm.

$i \in I$ [6] . Noting

$$\alpha^i_{hh'}(u) - \alpha^i_{h'h}(u) = \frac{1}{\overline{\omega} - \underline{\omega}}(f(\pi(h', u^{-i})) - f(\pi(h, u^{-i}))),$$

for (2.4) we have

$$\dot{u}^i_h = \frac{r}{\overline{\omega} - \underline{\omega}}(f(\pi^i(e^i_h, u^{-i})) - \sum_{l \in S^i} f(\pi^i(e^i_l, u^{-i})))u^i_h. \tag{2.7}$$

(2.7) is a payoff monotonic dynamics and especially represents replicator dynamics in the case of $f(x) = x$.

The proportional imitation with switching rate $\dfrac{x^i_{h'} - x^i_h}{\overline{\omega} - \underline{\omega}}$ is optimal , i.e, it maximizes the increase in expected payoffs among all behavioral rules in the single-person sampling model (proposition 1 in Schlag(1998)).

Reinforcement learning. This is a model in which the review rate depends on the level of satisfaction obtained by the payoff (Börgers and Sarin (1997), Cabrales(2000),Gale et al.(1995) and section 4.3.3.2 in Veda-Redonde(1996)).

When the current state is $u(t)$ at time t , each h-individual compares the utility $f(\pi^i(e^i_h, u^{-i}(t)))$ obtainend from the payoff $\pi^i(e^i_h, u^{-i}(t))$ to some target level of satisfaction μ. If $f(\pi^i(e^i_h, u^{-i}(t))) \geq \mu$, he retains strategy h. Otherwise he randomly chooses a new starategy according to the current population state $u^i(t)$. Here the satisfaction level is uniformly distribued on $[\underline{\omega}, \overline{\omega}]$ with $\underline{\omega} < \min_{h,i,u} f(\pi(e^i_h, u^{-i}), \overline{\omega} < \max_{h,i,u} f(\pi(e^i_h, u^{-i})$, and independent across individual and time.

For the reason why the satisfaction level is random, there two possible interpretations. One is that the mood of an individual (whether she is ambitious or not) is randomly determined (section 4.3.3.2 in Veda-Redonde[ ]). The other is that the satisfaction level is actually fixed while the utility obtainend from the payoff is perturbated by random shock.

In this scenario the average review rate and the choice probability are taken as $r^i_h(u) = \dfrac{\overline{\omega} - f(\pi(e^i_h, u^{-i}))}{\overline{\omega} - \underline{\omega}}$ and $p^i_h(u) = u^i_h$. Then we have (2.7) again. The imitation scheme of this examle is of class II and satisfies all assumptions in subsection 2.2.

[6] Since $\alpha^i_{hh'}(u)$ is a continuous function on Θ, (B2:i) is satisfied from the compactness of Θ.

3 Mckean process

In this section we construct a Mckean process on S perfectly describing imitational behavior such that the probability distribution at time t itself is represented by imitation dynamics.This is done in a martingale formulation [7] , following Shiga and Tanaka(1985). Further we prove the law of large numbers and the propagation of chaos for Λ-particle system. First we construct a continuous-time Markov chain in a martingale formulation.

Given r^i, p^i , set $m_i \times m_i$-matrix valued function q^i on Θ by

$$q^i(h,h';u) = \begin{matrix} r^i_h(u)p^i_{hh'}(u), & h' \neq h \\ 0, & h' = h \end{matrix}.$$

Then define bounded linear operators $Q^i(u), u \in \Theta$ on $\mathsf{B}(S^i)$ and $Q(u), u \in \Theta$ on $\mathsf{B}(S)$ by

$$Q^i(u)\phi(h) = \sum_{l\in S_i} q^i(h,l;u)(\phi(l)-\phi(h)), h \in S^i, \phi \in \mathsf{B}(S^i),$$
$$Q(u)\varphi(h^1,\dots,h^n) = \textstyle\sum_{i\in I}\sum_{l\in S_i} q^i(h^i,l;u)\,\triangle^i_l\,\varphi(h^1,\dots,h^n), \varphi \in \mathsf{B}(S),$$

where $\triangle^i_l\varphi(h^1,\dots,h^n) = \varphi(h^1,\dots,\underset{i-th}{l},\dots,h^n) - \varphi(h^1,\dots,h^n)$, and $\mathsf{B}(S^i)$ ($\mathsf{B}(S)$) is the Banach space of all bounded functions on S^i (resp. S) equipped with the supremum norm $\|\cdot\|$.

Let $X^i(X)$ be a stochastic process defined on a probability space $(\Omega,\mathcal{F},\mathrm{P})$ of that sample paths are $S^i(resp.\ S)$-valued right-continuous step functions, and set $\mathcal{F}^{X^i}_t = \sigma(X^i(s); s \le t)(resp. \mathcal{F}^X_t = \sigma(X(s); s \le t))$.

Definition 1. For Θ- valued measurable function $v(t)$ and $\overline{v}^i \in \triangle^i$, X^i is a *solution of the martingale problem (MP)* for $(\{Q^i(v(t))\},\overline{v})$ if X^i satisfies

$$\phi(X^i(t)) - \int_0^t Q^i(v(s))\phi(X^i(s))ds\ is\ a\ \mathcal{F}^{X^i}_t - martingale\ for\ any\ \phi \in \mathsf{B}(S^i), \tag{3.1}$$
$$\mathcal{L}(X^i(0)) = \overline{v}^i,$$

where $\mathcal{L}(X^i(0))$ stands for the probability law of $(X^i(0))$.

[7]For details of a martingale problem, see Ethier and Kurtz (1985).

Definition 2. For a S- valued measurable function $v(t)$ and a $\overline{v}$, X is a *solution of the martingale problem (MP)* for $(\{Q(v(t))\}, \overline{v})$ if X satisfies

$$\varphi(X(t)) - \int_0^t Q(v(s))\varphi(X(s))ds \ is \ a \ \mathcal{F}_t^X - martingale \ for \ any \ \varphi \in \mathsf{B}(S), \tag{3.2}$$

$$\mathcal{L}(X(0)) = \overline{v}.$$

In the above definitions $\{Q^i(v(t))\}(\{Q(v(t))\})$ is called a *generator*, and $X^i(X)(X(t)$ is said to be *generated* by $\{Q^i(v(t))\}(resp. \ \{Q(v(t))\})$. A solution of MP for $(\{Q^i(v(t))\}, \overline{v^i})$ is a Markov process on S^i of that transition rate from l to h at time t is given by $q^i(l, h; v(t))$, and a solution of MP for $(\{Q(v(t))\}, \overline{v})$ is composed of mutually independent solution of MP for $(\{Q^i(v(t))\}, \overline{v^i}), i \in I$.

For any fixed $c^i > C^i_{r1}$, define transition probability matrix $P^i(u), u \in \Theta$ on S^i by

$$P^i_{hh'}(u) = \begin{cases} \frac{1}{c^i} r^i_h(u) p^i_{hh'}(u), & h' \neq h \\ 1 - \sum_{l \neq h} \frac{1}{c^i} r^i_h(u) p^i_{hh'}(u), & h' = h \end{cases}. \tag{3.3}$$

Then a solution of the MP is simply constructed as follows. Let $X^i(0)$ be independent S^i-valued random variable with the distribution $\overline{v^i}$,and $N^i(t)$ be a Poisson process on $\{0, 1, \dots\}$ with intensity c^i independent of $X^i(0)$, denoting by σ^i_n the n-th jump time of $N^i(t)$. Set $X^i(t) = X^i(0)$ for $\sigma^i_0 = 0 \leq t < \sigma^i_1$. At $t = \sigma^i_1$, $X(\sigma^i_1)$ is randomly chosen according to the transition law $P^i(v(t))$, and set $X^i(t) = X^i(\sigma^i_1)$ for $\sigma^i_1 \leq t < \sigma^i_2$. Repeating this prcedure we obtain S^i- valued process $X^i(t)$, which is a solution of the MP $(\{Q^i(v(t))\}, \overline{v^i})$.

Let $X_i(t), i \in I$ be mutually independent S^i-valued Markov processes constructed in the above, and set $X(t) = (X_1(t), \dots, X_n(t))$. Then it is shown that $X(t)$ is a unique solution of MP for $(\{Q(v(t))\}, \overline{v})$ (see the proof of proposition 1(i)).

Now we give the definition of a Mckean process. It is a Markov process that moves under the influence of the distribution of itself at each time, and therefore it represents a stochatic phenomenon with interactions.

Definition 3. For $\overline{u} \in \Theta$, X is a *Mckean process correspnding to* (abbreviated as c.t.) ($\{Q(v); v \in \Theta\}$) with $\mathcal{L}(X(0)) = \bar{u}$ if X is a solution of the MP,

$$\begin{aligned} &\varphi(X(t)) - \int_0^t Q(u(s))\varphi(X(s))ds \; is \; a \; \mathcal{F}_t^X - martingale \; for \; any \; \varphi \in \mathsf{B}(S), \\ &\mathcal{L}(X(t)) = u(t) \; and \; \mathcal{L}(X(0)) = \bar{u}. \end{aligned} \tag{3.4}$$

When $Q(v)$ is given by an imitation scheme $(r^i, p^i), i \in I$, we say a Mckean process is *associated with* an imitation scheme $(r^i, p^i), i \in I$. From now on, by a Mckean process we mean a Mckean process associated with an imitation scheme $(r^i, p^i), i \in I$.

In definition 3 a generator $\{Q(u(t))\}$ depends on $u(t)$, the marginal distribution of a Mckean process itself while a generator $\{Q(v(t))\}$ does on exogenously given $v(t)$ in definition 2. This difference characterizes a Mckean process as a stochastic process that represents a stochastic phenomenon with interactions.

If we substitute $\varphi = 1_h, h \in S^i$ in (3.4) and take the expectation, we have

$$\begin{aligned} &u_h^i(t) - \bar{u}_h^i \\ &= \int_0^t \sum_l [r_l^i(u(s))p_{lh}^i(u(s))u_l^i(s) - r_h^i(u(s))u_h^i(s)ds, h \in S^i, i \in I. \end{aligned} \tag{3.5}$$

By the continuity of r^i and p^i, (3.5) is equivarent to

$$\dot{u}_h^i = \sum_{l \in S^i} r_l^i(u)p_{lh}^i(u)u_l^i - r_h^i(u)u_h^i, u_h^i(0) = \bar{u}_h^i, h \in S^i, i \in I.$$

This is the imitation dynamics (2.4) given by Weibull(1995) and Björnersted and Weibull (1995). The unique existence of a solution for (2.4) is guaranteed by the Lipschitz continuity of $r^i, p^i, i \in I$. So the marginal distribution of the Mckean process is a unique solution of the imitation dynamics (2.4).

Now we consider an Λ-particle sysytem, which is a Markov process $X^{(\Lambda)} = (X_1(t), \ldots, X_\Lambda(t))$ on $S^{\otimes\Lambda}$ generated by the following operater based on $Q(u)$:

$$Q^{(\Lambda)}\Phi(H_1, \ldots, H_\Lambda) = \sum_{\lambda=1}^{\Lambda} Q_\lambda(\frac{1}{\Lambda}\sum_{\lambda=1}^{\Lambda} H_\lambda^1, \ldots, \frac{1}{\Lambda}\sum_{\lambda=1}^{\Lambda} H_\lambda^n)\Phi(H_1, \ldots, H_\Lambda), \Phi \in \mathsf{B}(S^{\otimes\Lambda}),$$

where $Q_\lambda(\frac{1}{\Lambda}\sum_{\lambda=1}^{\Lambda} H_\lambda^1, \ldots, \frac{1}{\Lambda}\sum_{\lambda=1}^{\Lambda} H_\lambda^n)$ stands for the operation of $Q(\frac{1}{\Lambda}\sum_{\lambda=1}^{\Lambda} H_\lambda^1, \ldots, \frac{1}{\Lambda}\sum_{\lambda=1}^{\Lambda} H_\lambda^n)$ w.r.t. H_λ.

Let W be the set of all S- valued right continuous step functions with left limits, and define σ- field of W by $\mathcal{F}_t = \sigma(w(s); s \geq t)$ and $\mathcal{F} = \vee_{t\geq 0} \mathcal{F}_t$ for coordinate process $w(t), w \in W$. We denote by μ and $P^{(\Lambda)}$ the probability measures on $(W, \mathcal{F})$ induced by the Mckean process and on $(W^\Lambda, \mathcal{F}^\Lambda)$ induced by $X^{(\Lambda)}$ respectively.

Then $\{P^{(\Lambda)}, \Lambda > 1\}$ is said to be *μ-chaotic* if

$$\lim_{\Lambda\to\infty} < P^{(\Lambda)}, \varphi_1 \otimes \cdots \otimes \varphi_M \otimes 1 \otimes \cdots \otimes 1 >= \prod_{\lambda=1}^{M} < \mu, \varphi_\lambda > \; for\ any\ \varphi_\lambda \in \mathsf{B}(W). \quad (3.6)$$

(3.6) implies $P^{(\Lambda)}$ weakly converges to $\mu^{\otimes\Lambda}$, and is equivarent to the law of numbers for $U^{(\Lambda)}$,i.e., $\lim_{\Lambda\to\infty} U^{(\Lambda)} = \mu$ in probability (Snitzman(1984)) and Tanaka (1983)), where $U^{(\Lambda)} = \dfrac{1}{\Lambda}\sum_{\lambda=1}^{\Lambda} \delta_{X_\lambda}$.

The following proposition is proved by an application of a standard argument to a multiple state-space case (see Shiga and Tanaka (1985), Sznitman (1984) and Tanaka (1982)).

Proposition 1. Given $r^i, p^i, i \in I$, the following statements hold.
(i) For any $\bar{u} = \bar{u}^1 \otimes \cdots \otimes \bar{u}^n \in \Theta$ there exists a unique Mckean process X c.t. $\{Q^v; v \in \Theta\}$ with $\mathcal{L}(X(0)) = \bar{u} \in \Theta$ such that $u^i(t) = \mathcal{L}(X^i(t))$ is a unique solution of (2.4).
(ii)If $\mathcal{L}(X^\Lambda(0)) = \bar{u}^{\otimes\Lambda}$ for $\bar{u} \in \Theta$, then for any $\varepsilon > 0$ and $T(0 < T < \infty)$ $P^{(\Lambda)}$ is μ-chaotic, and

$$\lim_{\Lambda\to\infty} P(\sup_{0\leq t\leq T} ||U^{(\Lambda)}(t) - u(t)|| > \varepsilon) = 0, \quad (3.7)$$

where $u(t) = \mathcal{L}(w(t); \mu)$.

Proposition 1 says that Λ-particle system generated by $Q^{(\Lambda)}$ converges in law to the infinite direct product of the Mckean process. This fact enables us to study each individual's imitational behavior in populations with countable individuals based on the analysis of the Mckean process as already stated.

From the proof of proposition 1(i), we note that the Mckean process is constructed based on review rates $r^i(u(t))$ and choice probabilities $p^i(u(t)), i \in I$ with $u(t)$, the unique solution of (2.4). Further, we remark that by replacing r^i by αr^i

for all $i \in I$ with any postive constant α , we have a Mckean process which is different from the original only in time scale. So the difference by a constant multiple factor of the review rates does not have any influence on our results.

4 Elimination of strategy

One of the most important problems concering dynamics for imitational behavior is whether it eliminates suboptimal strategies in the long run. In this section we study the elimination of a strategy on the Mckean process.

First we state a proposition for the solution of the Markov process generated by $\{Q(v(t)\}$, which gives sufficient conditions for a strategy to vanish and not to vanish with probability one. Then as a corollary of the proposition, we present our first main result that a necessary and sufficient condition for a stochastic path of the Mckean process to converge to a pure strategy is that the pure strategy is a unique best reply in the limit state.

Proposition 2. For $k \in S^i$, the following statements hold for X generated by $\{Q(v(t)\}$.
(i)Under (B1:i) and (B2:i), if $\overline{\lim_{t\to\infty}} v^i_k(t)t^\alpha < +\infty$ with $\alpha > 1$, then $P(\lim_{t\to\infty} 1_k(X^i(t)) = 0) = 1$.
(ii) Under (B2:i), if $\underline{\lim_{t\to\infty}} \frac{v^i_k(t)}{g(t)} > 0$ with positive decreasing function g such that $\sum_{n=1}^{\infty} g(bn) = +\infty$ for any constant $b > 0$, then $P(\lim_{t\to\infty} 1_{k_i}(X^i(t)) = 0) = 0$.

Proposition 2.(i) implies that if the choice probability for strategy k vanishes sufficiently fast, each individual stops to take strategy k after a finite time interval. On the contrary, proposition 2.(ii) implies that even when the choice probability converges to zero, if the speed of the convergence is not so fast, each individual never stops to take strategy k.

The following corollary is proved by use of proposition 2.(i) and a result of Samuelson and Zhang (1992) about payoff monotonic dynamics. For example, the marginal distribution of the Mckean process associated with an imitation scheme of class II such that $\{-r^i\}$ is payoff monotonic, represents payoff monotonic dynamics. Then, due to the corollary, each individual stops to take suboptimal strategy k dominated by some pure strategy after a finite time interval in the imitation scheme.

Corollary 1. Suppose that the marginal distribution of a Mckean process constitutes a payoff monotone dynamics. If pure strategy k is stricly iteratively dominated by another pure strategy in S^i, then for the Mckean process $P(\lim_{t\to\infty} 1_k(X(t)) = 0) = 1$ for any initial state $\bar{u} \in int(\Theta)$.

The following main result is also proved as a corollary of proposition 2. Roughly speaking, in proposition3(ii) the assumption that $\lim_{u\to u^*} \frac{G_l^j(u)u_l^j}{G_h^i(u)u_h^i}$ is finite implies $|u_l^j(t) - u_l^{*j}|$ is proportional to $u_h^i(t)$ when $u(t)$ approaches to u^*. (i) and (ii) together say that under this technical assumption each individual in population i goes to fix on strategy k if and only if strategy k is a unique best reply to the limit state. Then (iii) immediately follows from (i) and (ii).

Proposition 3. Suppose that the marginal distribution $u(t)$ of the Mckean process represents a payoff monotonic dynamics and that $\lim_{t\to\infty} u(t) = u^*$ with $u^{*i} = e_k^i$. Then the following statements hold.
(i)If $k \in S^i$ is a unique best reply against u^{*-i}, then $P(\lim_{t\to\infty} X^i(t) = k) = 1$.
(ii)If for any $l \in S^j$, $j \neq i \in I$ there exists $h_l^j \neq k \in S^i$ that is a best reply against u^{*-i} and $\lim_{u\to u^*} \frac{\dot{u}_l^j(u)}{\dot{u}_{h_l^j}^i(u)} = \lim_{u\to u^*} \frac{G_l^j(u)u_l^j}{G_{h_l^j}^i(u)u_{h_l^j}^i}$ are finite, then $P(\lim_{t\to\infty} 1_{h_2}^i(X^i(t)) = 0) = 0$ for some $h_2 \neq k \in S^i$ that is a best reply against u^{*-i}.
(iii)For $n = 1$, i.e., a pairwise contest case in a single population, $P(\lim_{t\to\infty} X(t) = k) = 1$ if and only if (k, k) is a strict Nash equilibrium.

Consider a case where the marginal distribution of the Mckean process represents a replicator dynamics in a single population. Let k be an evolutionarily stable strategy that does not constitute a strict Nash equilibrium.

As is well known, in a replicator dynamics any evolutionarily stable strategy is asymptotically stable. Therefore, if the initial population state belongs to the attractive domain of e_k, the population state goes to e_k by the imitational behavior of individuals. But due to proposition 3(iii), each individual never fixes on strategy k in spite of our expectation that he surely does. Unfortunately the evolutionary stability does not insure the convergence of strategy in an individual-wise.

For example, when we model imitational behavior in our society by a replicator dynamics, we apt to think each member in our society goes to take a pure strategy that is evolutionarily stable on the basis of the convergence of trajectory to the strategy. But proposition 3(iii) says that this reasoning is false.

5 ergodicity

As is well known, ergodic theorem is generally formulated as the time average of a function of a stationary process converges to the state-space average of that. In our case by Birkhoff's individual ergodic theorem (theorem 3 in Skorohod (1989)), $\lim_{t\to\infty} \frac{1}{t}\int_0^t f(X(s))ds = < \bar{u}, f >$ a.s. for a Mckean process associated with an imitation scheme in class II, starting from any stationary state $\bar{u} \in int(\Theta)$. This is equivarent to that the average sojourn time at any strategy converges to the weight of the strategy at $\bar{u}$ for almost all paths.

From the viewpoint of imitational behavior, we can interpret this fact as follows. For an imitation scheme in class II, the imitation scheme of all individuals in every population are the same at any stationary state, i.e., $r^i_h(\bar{u}) = r^i_l(\bar{u}), h, l, \in S^i$ and $p^i_h(\bar{u}) = p^i_l(\bar{u}), h, l, \in S^i$ for every $i \in I$. Therefore, all individual's time average of holding each strategy are equalized.

In this section we extend this result to a case where the marginal distribution of a Mckean process staring from a state that may be non-stationary converges to some state. The next proposition is our second main result.

Proposition 4. For every $i \in I$, assume that (B3:i) holds and that the marginal distribution $u(t)$ of a Mckean process converge to $u^* \in \Theta$ such that $r^i_h(u^*)$ is independent of h, i.e.,$r^{i*} = r^i_l(u^*), l \in S^i$ for some $r^{i*} > 0$. Then for the Mckean process,

$$P(\lim_{t\to\infty}\frac{1}{t}\int_0^t \varphi(X(s))ds =< \varphi, u_{h^1}^{1*}\cdots u_{h^n}^{n*} >) = 1, \varphi \in \mathrm{B}(S), \tag{5.1}$$

where $< \Phi, u >$ stands for the expectation of Φ w.r.t. u.

In particular

$$P(\lim_{t\to\infty}\frac{1}{t}\int_0^t 1_H(X(s))ds = u_{h^1}^{1*}\cdots u_{h^n}^{n*}) = 1, H = (h^1,\ldots,h^n) \in S, \tag{5.2}$$

where $1_H = 1_{h^1}\otimes\cdots\otimes 1_{h^n}$,

$$P(\lim_{t\to\infty}\frac{1}{t}\int_0^t \pi^i(X(s))ds = \pi^i(u^*)) = 1, i \in I. \tag{5.3}$$

In a case where a Mckean process begins from a stationary state, proposition 4 is reduced to Birkhoff's individual ergodic theorem. So proposition 4 is an extension of the theorem. In a case where an imitation scheme is of class I with $p_{lh}^i = p_h^i, l, h \in S^i, i \in I$ or of class II with $u^* \in int(\Theta)$, the assumptions are satisfied, and proposition 4 can be applied.

Under the assumption of proposition 4, when $u(t)$ approaches to u^*, the review rate of all individual converge to the same rate, and they goes to behave in the same way in every populaton. Then the behavior of all individuals in time average become equalized in each populaton, independently of other population. The time average of holding each strategy coincides the weight of it at the limit state.

Consequently, the time average of the payoff realized for each individual is given by the average payoff at the limit state as is seen in (5.3). When $u(t)$ goes to u^*, of course $\frac{1}{t}\int_0^t \pi^i(u(s))ds$ does to $\pi^i(u^*)$, that is, the time average of the average payoff in each population at time t converges to that at the limit state. (5.3) shows that not only as for the average payoff in each population, but also as for each individual's realized payoff in each population, the time average converges to the average payoff at the limit state.

The following counter example shows that without assumption (B3:i) that p_{lh}^i does not depend on l, a case where a Mckean process X is a reducible Markov chain is allowed, and then the ergodicity does not hold. Assumption (B3:i) implicitly excludes such a case.

Counter example. Set $S = \{1, 2, 3, 4\}$ and consider the Mckean process associated by the following review rate function and choice probability function:

$$r_h(u) = 1, h \in S,$$
$$p_{hl}(u) = \begin{matrix} \begin{cases} 2u_l, & l = 1, 2 \\ 0, & l = 3, 4 \end{cases} & if\ h = 1, 2 \\ \begin{cases} 0, & l = 1, 2 \\ 0, 2u_l & l = 3, 4 \end{cases} & if\ h = 3, 4 \end{matrix}.$$

Then (2.4) turns into

$$\dot{u}_h = \begin{matrix} 2(u_1 + u_2)u_h - u_h, & h = 1, 2 \\ 2(u_3 + u_4)u_h - u_h, & h = 3, 4 \end{matrix}.$$

This shows any $\bar{u} \in \triangle$ with $\bar{u}_1 + \bar{u}_2 = \frac{1}{2}$ is a stationary state. Hence, by lemma 2 in subsection A.3, for the Mckean process starting from such a stationary state $\bar{u}$,

$$P(\lim_{t\to\infty} \frac{1}{t} \int_0^t 1_l(X(s))ds = \bar{u}_l, l = 1, 2|X(0) = h) = 1, h = 1, 2,$$
$$P(\lim_{t\to\infty} \frac{1}{t} \int_0^t 1_l(X(s))ds = \bar{u}_l, l = 3, 4|X(0) = h) = 1, h = 3, 4.$$

So we have

$$P(\lim_{t\to\infty} \frac{1}{t} \int_0^t 1_l(X(s))ds = \bar{u}_l, l = 1, 2) = P(X(0) = 1, 2) = \frac{1}{2},$$

which shows (5.2) does not hold in this case.

6 Applications

In this section we demonstrate how our results are applied cases where the marginal distribution of a Mckean process represents a replicator dynamics in 2-populations.

Consider the following imitation scheme for two populations in classs I. Let A and B be normalized payoff matrixes for population 1 and 2 :

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}.$$

Denote by $(x_1, x_2)((y_1, y_2))$ a element of $\triangle^1(resp.\ \triangle^2)$, and set review rate functions and choice probability functions as follows: $r_h^1(u) = |a_1|+|a_2|+1-\pi^1(e_h^1, u^2)$, $p_{lh}^1(u) = u_h^1, h = 1, 2$ and $r_h^2(u) = |b_1| + |b_2| + 1 - \pi^2(u^1, e_h^2)$, $p_{lh}^2(u) = u_h^2, h = 1, 2.$ Then the marginal distribution of the Mackean process (X, Y) is a solution of the following replicator dynamics in the two-populations.

$$\dot{x}_1 = (a_1 y_1 - a_2 y_2) x_1 x_2, \dot{x}_2 = -\dot{x}_1, \tag{6.1a}$$

$$\dot{y}_1 = (b_1 x_1 - b_2 x_2) y_1 y_2, \dot{y}_2 = -\dot{y}_1. \tag{6.1b}$$

We analyze the pathwise behavior of the Mckean process (X, Y) when the marginal distribution converges to limit states in two cases.

Case of $\mathbf{a_1 = 0, a_2 > 0, b_1 > 0}$ and $\mathbf{b_2 = 0}$. By (6.1) $x_1(y_1)$ monotonicaly decreases (resp. increases) from any $(\bar{x}, \bar{y}) \in \Theta$ along $(\frac{x_2}{\bar{x}_2})^{b_1} = (\frac{y_1}{\bar{y}_1})^{a_2}$, and the limit state (x_1^*, y_1^*) is such that

$$(x_1^*, y_1^*) = \begin{cases} (1 - \dfrac{1 - \bar{x}_1}{\bar{y}_1^{a_2/b_1}}, 1), & (1 - \bar{x}_1)^{b_1} < \bar{y}_1^{a_2} \\ (0, 1), & (1 - \bar{x}_1)^{b_1} = \bar{y}_1^{a_2}. \\ (0, \dfrac{\bar{y}_1}{(1 - \bar{x}_1)^{b_1/a_2}}), & (1 - \bar{x}_1)^{b_1} > \bar{y}_1^{a_2} \end{cases}$$

When $(x_1^*, y_1^*) = (1 - \frac{1 - \bar{x}_1}{\bar{y}_1^{a_2/b_1}}, 1)$, strategy 1 and 2 are indifferent for population 1 while and strategy 1 is a unique best reply for population 2.

Then, noting $r_1^1(1 - \frac{1 - \bar{x}_1}{\bar{y}_1^{a_2/b_1}}, 1) = r_2^1(1 - \frac{1 - \bar{x}_1}{\bar{y}_1^{a_2/b_1}}, 1) = a_2 + 1$, by proposition 3 and 4 we have

$$P(\lim_{t\to\infty} \frac{1}{t} \int_0^t 1_1(X(s))ds = 1 - \frac{1 - \bar{x}_1}{\bar{y}_1^{a_2/b_1}}) = 1, P(\lim_{t\to\infty} Y(t) = 1) = 1,$$

if $(1-\bar{x}_1)^{b_1} < \bar{y}_1^{a_2}$. Similary if $(1-\bar{x}_1)^{b_1} > \bar{y}_1^{a_2}$,

$$P(\lim_{t\to\infty} X(t) = 2) = 1, P(\lim_{t\to\infty} \frac{1}{t}\int_0^t 1_1(Y(s))ds = \frac{\bar{y}_1}{(1-\bar{x}_1)^{b_1/a_2}}) = 1.$$

For the case of $(1-\bar{x}_1)^{b_1} = \bar{y}_1^{a_2}$, $(0,1)$ is not a strict Nash equilibrium,and $\lim_{(x_1,y_1)\to(0,1)} |\frac{\dot{y}_1}{\dot{x}_1}| = |\frac{b_1}{a_1}| < +\infty$. So by proposition 3(ii) and 4,

$$P(\lim_{t\to\infty} X(t) = 2) = P(\lim_{t\to\infty} Y(t) = 1) = 0,$$

$$P(\lim_{t\to\infty} \frac{1}{t}\int_0^t 1_2(X(s))ds = 1) = P(\lim_{t\to\infty} \frac{1}{t}\int_0^t 1_1(Y(s))ds = 1) = 1.$$

In the case where the initial population state on the curve $(1-\bar{x}_1)^{b_1} = \bar{y}_1^{a_2}$, the population state converges to $(0,1)$ while (X, Y) never does to $(2,1)$. But it holds that

$$P(\lim_{t\to\infty} \frac{1}{t}\int_0^t \pi^1(X(s), Y(s))ds = a_2, \lim_{t\to\infty} \frac{1}{t}\int_0^t \pi^2(X(s), Y(s))ds = b_1) = 1,$$

and all individuals equally gain the maximum time-average of payoff.

Case of $\mathbf{a_1 < 0, a_2 < 0, b_1 = 0}$ and $\mathbf{b_2 < 0}$. This case contains the simplified ulitimatum game of Gale et al.(1995) (also see section 4.8 in Vega-Redonde (1996)). As the previous case, the limit state (x_1^*, y_1^*) is such that

$$(x_1^*, y_1^*) = \begin{cases} (0,1), & \bar{x}_1^{-b_2} < (\frac{\bar{y}_1}{\hat{y}_1})^{-a_2}(\frac{\bar{y}_1}{\hat{y}_1})^{-a_1} \\ (1,\hat{y}_1), & \bar{x}_1^{-b_2} = (\frac{\bar{y}_1}{\hat{y}_1})^{-a_2}(\frac{\bar{y}_1}{\hat{y}_1})^{-a_1}, \\ (1,y_1^*), (0 < y_1^* < \hat{y}_1), & \bar{x}_1^{-b_2} > (\frac{\bar{y}_1}{\hat{y}_1})^{-a_2}(\frac{\bar{y}_1}{\hat{y}_1})^{-a_1} \end{cases}$$

where $\hat{y}_1 = \dfrac{a_2}{a_1+a_2}$.

Since $(0,1)$ is a strict Nash equilibrium, by proposition 3(i),

$P(\lim_{t\to\infty} X(t) = 2, \lim_{t\to\infty} Y(t) = 1) = 1$ for $\bar{x}_1^{-b_2} < (\frac{\bar{y}_1}{\hat{y}_1})^{-a_2}(\frac{\bar{y}_1}{\hat{y}_1})^{-a_1}$.

For $\bar{x}_1^{-b_2} = (\frac{\bar{y}_1}{\hat{y}_1})^{-a_2}(\frac{\bar{y}_1}{\hat{y}_1})^{-a_1}$, $x_1^* = 0, y_1^* = 1$ are not a unique best reply to each other [8]. By proposition 4,

$$P(\lim_{t\to\infty} \frac{1}{t}\int_0^t 1_1(X(s))ds = 1, \lim_{t\to\infty} \frac{1}{t}\int_0^t 1_1(Y(s))ds = \hat{y}_1) = 1.$$

[8]Since $\lim_{(x_1,y_1)\to(1,\hat{y}_1)} |\frac{\dot{y}_1}{\dot{x}_1}| = +\infty$, proposition 3(ii) can not be applied.

For $\bar{x}_1^{-b_2} > (\frac{\bar{y}_1}{\hat{y}_1})^{-a_2}(\frac{\bar{y}_1}{\hat{y}_1})^{-a_1}$, by proposition 3(i) and 4,

$$P(\lim_{t\to\infty} X(t) = 1) = 1, P(\lim_{t\to\infty} \frac{1}{t}\int_0^t 1_1(Y(s))ds = y_1^*) = 1.$$

7 Concluding remarks.

We have analyzed the imitational behavior of each individual in a multi-population model when the population state converges to an equilibrium state. Our approach is characterized by the following two points. One is that it is not to analyze the behavior of the population shares individuals, but to do the stochatic behavior of each individual itself. The other is that it has a generic frame so that it is applicable to braod classes of imitation or learning models.

We have shown that each individual does settle on a pure strategy in the long run if and only if the pure strategy is a best reply to the limit state. Moreover, all individuals' average holding time of each strategy becomes equalized to the weight of the strategy at the limit state because their schemes of imitational behavior goes to the same.

Here we remark that these results heavily depends on the implicit assumption that individuals are myopic and memoryless. In our model individuals choose their strategies depending only on the current population state. This leads the stochastic process representing the behavior of each individual ro having Markov property, which plays an essential role in the proof of the results. The results for a model with individuals being not myopic or having memory might become different from ours.

References

[1] Billingsley, P.,1968, Convergence of Probability Measures, John Wiley & Sons.

[2] Björnerstedt,J.,Schlag, K.H., 1996, On the Evolution of imitation Behavior,mimeo.

[3] Björnerstedt,J.,Weibull, J., 1996, Nash equilibrium and evolution by imitation, in:Arrow,K et al.,(Eds.), The Rational Foundations of Economic Behavior, Macmillan, London,155-171.

[4] Börgers,T., Sarin,R., 1997, Learning Through Reinforcement and Replicator Dynamics, Journal of Economic Theory 77,1-14.

[5] Börgers,T., Sarin,R., 2000, Naive Reinforcement Learning with Endogenous Aspirations, International Economic Review 41,921-950.

[6] Ethier,S.,Kurtz, T.,1985, Markov Processes, John Wiley & Sons.

[7] Gale,J.,Binmore,K.G.,Samuelson,L.,1995 Learning To Be Imperfect: The Ultimatum Game, Games and Economic Behavior 8, 56-90.

[8] Gihman,I.I., Skorohod,A.V., 1975, The Theory of Stochastic Processes II, Springer-Verlag.

[9] Hoel,P.G., Port,S.C., Stone, C.J., 1971, Introduction to Stochastic Process. Houghton Mifflin Company.

[10] Lindval,T., 1992, Lectures on the Couplimg Method, John Wiley & Sons.

[11] Samuelson,L.,Zhang,J.,1992, Evolutionary Stability in Asymmetric Games. Journal of Economic Theory 57,363-391.

[12] Schlag, K.H., 1998, Why imitate, and if so, how? A bounded ratinal approach to multi-armed bandits, Journal of Ecnomic Theory 78,130-156.

[13] Shiga,T., Tanaka, H.,1985, Central Limit Theorem for a System of Markovian Particles with Mean Field Interactions, Zeitschrift für Wahrscheinlichkeitsthorie und verwandte Gebiete 69, 439-459.

[14] Skorokhod, A.V.,1989, Asymptptic Methods in the Theory of Stochastic Differential Equations, American Mathematical Society.

[15] Sznitman, A.S.,1984, Nonlinear Reflecting Diffusion Process, and the Propagation of Chaos and Fluctuations Associated, Journal of Functional Analysis 56, 311-336.

[16] Tanabe, Y.,2001, Stochastic Property of Imitation Dynamics, discussion paper series No.2001-01, Faculty of Economics, Nagasaki-University.

[17] Tanaka,H.,1983, Some probabilistic problems in the spatially homogeneous Boltzmann equation of Maxwellian molecules in several dimensions, in:Kallianpur,G,(Eds.), Theory and application of Random Fields, Springer-Verlag,258-267.

[18] Weibull, J.,1995, Evolutionary Game Theory, MIT Press, Cambridge, MA.

[19] Vega-Redondo, F., 1996, Evolution, Games, and Economic Behaviour, Oxford-University Press.

A Appendix

A.1 Proofs for section 3

Proof of proposition 1(i). We prove the proposition in three steps. This is a simple extension of the proof of lemma 2 in Shiga and Tanaka (1985) to a multiple state-space case.

1st step. In the first step we show that $X(t)$ constructed by (3.3) is a unique solution of MP for $(\{Q(v(t))\}, \overline{v})$.

For $w^i, w \in W$ and $h \in S^i$, set

$$N(t, h; w^i) = \sum_{s \le t} 1(w^i(s) = h, w^i(s) \neq w^i(s-)),$$
$$\tilde{N}(t, h; w^i) = N(t, h; w^i) - \int_0^t q^i(w^i(s), h; v(s))ds.$$

Then $\tilde{N}(t, l; w^i)$ is a P^X-martingale by lemma 3 in Shiga and Tanaka (1985).

Since $w^i(t), i \in I$ have no common jumps, it holds that

$$\varphi(w(t)) - \varphi(w(0)) = \sum_{i \in I} \int_0^t \sum_{l \in S^i} \triangle_l^i \varphi(w(s-))N(ds, l; w^i).$$

Then we have for any $\varphi \in \mathsf{B}(S)$

$$\varphi(w(t)) - \varphi(w(0)) - \int_0^t Q(v(s))\varphi(w(s))ds$$
$$= \sum_{i \in I} \int_0^t \sum_{l \in S^i} \triangle_l^i \varphi(w(s-)))\tilde{N}(ds, l; w^i)ds,$$

and we conclude $X(t)$ is a solution of MP $(\{Q(v(t))\}, \overline{v})$.

Next, we show the uniqueness of the MP. For any solution X of the MP,

$$\begin{aligned}&E^X[\varphi(w(t)); A_s] - E^X[\varphi(w(s)); A_s]\\&= \int_s^t \sum_{i\in I} \sum_{h^1,\ldots,h^n} \sum_{l\neq h^i}[q^i(h^i, l; v(s_1))(\varphi(h^1, \ldots, l, \ldots, h^n) - \varphi(h^1, \ldots, h^n))\\&\cdot P^X((w(s_1) = (h^1, \ldots, h^i, \ldots, h^n)) \cap A_s)]ds_1, A_s \in \mathcal{F}_s.\end{aligned}$$

If $E^X[\varphi(w(s)); A_s] = E^Y[\varphi(w(s)); A_s]$ for any pair of solutions (X, Y) , then

$$\begin{aligned}&|E^X[\varphi(w(t)); A_s] - E^Y[\varphi(w(t)); A_s]|\\&\leq \int_s^t 2(\textstyle\sum_i C^i_{r1}) \parallel \varphi \parallel \cdot \parallel P^X((w(s_1) \in \cdot) \cap A_s) - P^Y((w(s_1) \in \cdot) \cap A_s) \parallel_{var} ds_1,\end{aligned}$$

where $\parallel \mu \parallel_{var}$ stands for the total variation norm of set function μ on the field of all subsets of S ,i.e., $\parallel \mu \parallel_{var} = \sum_{H\in S} |\mu(\{H\})|$.

This imples

$$\begin{aligned}&\parallel P^X((w(t) \in \cdot) \cap A_s) - P^Y((w(t) \in \cdot) \cap A_s) \parallel_{var}\\&\leq \int_s^t 2(\textstyle\sum_i C^i_{r1}) \parallel \cdot \parallel P^X((w(s_1) \in \cdot) \cap A_s) - P^Y((w(s_1) \in \cdot) \cap A_s) \parallel_{var} ds_1,\end{aligned} \tag{A.1}$$

Hence by Gronwall's inequality we have

$$\parallel P^X((w(s_1) \in \cdot) \cap A_s) - P^Y((w(s_1) \in \cdot) \cap A_s) \parallel_{var} = 0, s < \forall s_1 < t,$$

which shows the uniqueness of the MP for $(\{Q(v(t))\}, \overline{v})$.

2nd step. We show the following equation has a unique Θ-valued solution for any $u \in \Theta$.

$$\begin{aligned}&< u(t), \varphi > - < u(0), \varphi > = \int_0^t < u(s), Q(u(s))\varphi > ds, \varphi \in \mathrm{B}(S),\\&u(0) = u,\end{aligned} \tag{A.2}$$

By linearity (A.2) is equivarent to

$$\begin{aligned}&u(t)_{k^1\ldots k^n} - u(0)_{k^1\ldots k^n}\\&= \int_0^t \sum_{i\in I} \sum_{l\neq k^i}[q^i(l, k_i; u(s))u(s)_{k^1,\ldots,l,\ldots,k^n} - q^i(k_i, l; u(s))u(s)_{k_1,\ldots,k_i,\ldots,k_n}]ds,\\&K = (k^1, \ldots, k^n) \in S,\end{aligned}$$

where $u_{k^1\ldots k^n}(t) = u(t)(\{k^1,\ldots,k^n\})$.

This is a $\prod_i m^i$-dimensional integral equation for $u(t)_{k_1\ldots k_n}, k_1 \in S^1,\ldots,k_n \in S^n$ which has a unique solution since $q^i(h,k;\cdot), h,k \in S^i, i \in I$ are Lipschitz continuous. Further it is easily verified that $u(t)_{k_1\ldots k_n}, k_1 \in S^1,\ldots,k_n \in S^n$ constitute a probability. So A.2 has a unique Θ- valued solution.

Similary for any Θ-valued measurable function $v(t)$ and $u \in \Theta$, we have a unique Θ-valued solution for

$$< u(t),\varphi > - < u(0),\varphi >= \int_0^t < u(s), Q(v(s))\varphi > ds, \varphi \in \mathsf{B}(S),$$
$$u(0) = u, \tag{A.3}$$

3rd step. Let $u(t)$ be the unique solution of A.2 for $u = \overline{u}^1\otimes\cdots\otimes\overline{u}^n$ and X be the solution of MP for $(\{Q(u(t))\},\overline{u})$. Then $\tilde{u}(t) = \mathcal{L}(X(t))$ solves A.3 with $v(t) = u(t)$. This means $u(t) = \tilde{u}(t)$ and X is a Mckean process c.t. $(\{Q(v); v \in \Theta\})$ with $\mathcal{L}(X(0)) = \overline{u}$ because of the uniqueness of the solution of A.3. The uniqueness of the Mckean process is also proved from that of A.3 and MP for $(\{Q(u(t))\},u)$. □

Proof of proposition 1(ii). The proof proceeds in a fairly standard way as Sznitman (1984) and Tanaka (1983). Let W' be the space of Θ-valued right continuous paths with left limits. Then $U^{(\Lambda)}(t)$ is a Markov process with sample path in W'. Denote by $P'^{(\Lambda)}$ the probability measure on W' induced by $U^{(\Lambda)}(t)$. We prove the law of large numbers for $U^{(\Lambda)}$ (3.7), and then (3.6) easily follows.

First we show $\{P'^{(\Lambda)}\}$ is tight. Noting

$$U^{(\Lambda)}(t) = \left(\frac{1}{\Lambda}\sum_\lambda 1_{e^1_1}(w^1_\lambda(t)),\ldots,\frac{1}{\Lambda}\sum_\lambda 1_{e^i_h}(w^i_\lambda(t)),\ldots,\frac{1}{\Lambda}\sum_\lambda 1_{e^n_{m^n}}(w^n_\lambda(t))\right),$$

and

$$1_{e^i_h}(w^i_\lambda(t_2)) - 1_{e^i_h}(w^i_\lambda(t_1)) = \int_{t_1}^{t_2}\sum_{l\in S^i}(1_{e^i_h}(l) - 1_{e^i_h}(w^i_\lambda(s))N(ds,l;w^i_\lambda),$$

we have

$$
\begin{aligned}
&E^{(\Lambda)}[||U^{(\Lambda)}(t_2)-U^{(\Lambda)}(t_1)||^2|\mathcal{F}^{\Lambda}] \\
&=\frac{1}{\Lambda^2}E^{(\Lambda)}[\sum_i\sum_{h\in S^i}(\sum_\lambda\int_{t_1}^{t_2}\sum_{l\in S^i}(1_{e_h^i}(l)-1_{e_h^i}(w_\lambda^i(s))N(ds,l;w_\lambda^i))^2|\mathcal{F}^{\Lambda}] \\
&\le E^{(\Lambda)}[\sum_i\sum_{h\in S^i}(\sum_\lambda\int_{t_1}^{t_2}\sum_{l\in S^i}N(ds,l;w_\lambda^i)^2|\mathcal{F}^{\Lambda}] \qquad (A.4)
\end{aligned}
$$

Let $N_\lambda^i, i\in I, 1\le\lambda\le\Lambda$ be mutually independent Poisson processes with intensity c^i independent of $X^\lambda(0), 1\le\lambda\le\Lambda$. Then $X^{(\Lambda)}(t)$ is constructed by the jump time of $N_\lambda^i, i\in I$ and transition laws $(P^i_{X_\lambda^i(t)l}(U^{(\Lambda)}(t)))_{l\in S^i}, i\in I$. Thereore, since it holds that

$$
\begin{aligned}
&E^{(\Lambda)}[(\int_{t_1}^{t_2}\sum_{l\in S^i}N(ds,l;w_\lambda^i))^2|\mathcal{F}^{\Lambda}]\le c^i|t_2-t_1|(c^i|t_2-t_1|+1), \\
&E^{(\Lambda)}[(\int_{t_1}^{t_2}\sum_{l\in S^i}N(ds,l;w_\lambda^i))(\int_{t_1}^{t_2}\sum_{l\in S^i}N(ds,l;w_\kappa^i))|\mathcal{F}^{\Lambda}]\le c^{i2}|t_2-t_1|^2,
\end{aligned}
$$

by (A.4) we have

$$
E^{(\Lambda)}[||U^{(\Lambda)}(t_2)-U^{(\Lambda)}(t_1)||^2|\mathcal{F}^{\Lambda}]\le C(T)|t_2-t_1|, 0\le t_1\le t_2\le T,
$$

where $C(T)$ is a positive constant depending on T.

From this we have

$$
E'^{(\Lambda)}[||w(t_1)-w(t_2)||^2||w(t_2)-w(t_3)||^2]\le C^2(T)|t_1-t_3|^2, 0\le t_1\le t_2\le t_3\le T, \quad (A.5)
$$

which shows the tightness of $\{P'^{(\Lambda)}\}$ (theorem 15.6 in Billingsley(1968)).

Next we show for any limit point $P'^{(\infty)}$ of $\{P'^{(\Lambda)}\}$

$$
\begin{aligned}
&F(w) \\
&=<w(t),\varphi>-<w(0),\varphi>-\int_0^t<w(s),Q(w(s))\varphi>ds \\
&=0, P'^{(\infty)}-a.s., \varphi\in\mathrm{B}(S). \qquad (A.6)
\end{aligned}
$$

Let $\{P'^{(\Lambda_l)}\}$ be a subsequence converging to $P'^{(\infty)}$ and set

$$
M_\lambda(t)=\varphi(w_\lambda(t))-\varphi(w_\lambda(0))-\int_0^t<w_\lambda(s),Q(\frac{1}{\Lambda_l}\sum_\kappa w_\kappa(s))\varphi>ds.
$$

Then M_λ is a $P'^{(\Lambda_l)}$-martingale, and

$$< M_\lambda, M_\kappa >= 0, \lambda \neq \kappa. \tag{A.7}$$

Because

$$M_\lambda(t) = \sum_{i\in I} \int_0^t \sum_{h\in S^i} \triangle_h^i \varphi(w_\lambda(s-)))\tilde{N}(ds, h; w_\lambda^i)ds,$$

where $\tilde{N}(t, h; w_\lambda^i) = N(t, h; w_\lambda^i) - \int_0^t q^i(w_\lambda^i(s), h; U^{(\Lambda_l)}(s))ds$, and w_λ^i and w_κ^j have no common jumps for $\lambda \neq \kappa$.

By using (A.7) we have

$$E'^{(\infty)}F^2 = \lim E^{(\Lambda_l)} \frac{1}{\Lambda_l^2} |\sum_\lambda M_\lambda(t)|^2 = \lim \frac{1}{\Lambda_l} E^{(\Lambda_l)} M_1(t) = 0,$$

which imples (A.6).

Since $w(0) = \bar{u}$ by the law of large numbers, from the uniqueness of solution of (A.2) we have $w(t) = u(t), 0 \leq t \leq T, P'^{(\infty)}$-a.s., i.e., $P'^{(\infty)} = \delta_\mu$, which implies (3.7).

For the propagation of chaos, let $P_M^{(\Lambda)}$ be the marginal probability of $P^{(\Lambda)}$ on the first W^M. Then, we can have the tightness of $P_M^{(\Lambda)}$ as (A.5).

Further, by using (3.7) we can show inductively that for any $\varepsilon > 0$ there exists $\Lambda_n > 0$ and $C_{n-1}(T) > 0$ such that for $\Lambda > \Lambda_n$

$$\begin{aligned}
&|E_M^\Lambda[\Phi(\mathbf{w}(t_n)); A_{t_1} \cap \cdots \cap A_{t_{n-1}}] - E^{\mu^{\otimes M}}[\Phi(\mathbf{w}(t_n)); A_{t_1} \cap \cdots \cap A_{t_{n-1}}]| \\
&\leq C_{n-1}(T)||\Phi||\varepsilon + 2||\Phi||M(\sum_i C_{r_1}^i m^i) \\
&\cdot \int_{t_{n-1}}^{t_n} ||P_M^\Lambda((\mathbf{w}(s) \in \cdot) \cap A_{t_1} \cap \cdots \cap A_{t_{n-1}}) - \mu^{\otimes M}((\mathbf{w}(s) \in \cdot) \cap A_{t_1} \cap \cdots \cap A_{t_{n-1}})||_{var} ds, \\
&A_{t_1} \in \sigma(\mathbf{w}(t_1)), \ldots, A_{t_{n-1}} \in \sigma(\mathbf{w}(t_{n-1})), \Phi \in \mathsf{B}(S^{\otimes M}).
\end{aligned}$$

From this the weak convergence of the finite-dimensional distiributions of $P_M^{(\Lambda)}$ to those of $\mu^{\otimes M}$ follows. Thus, we have shown the weak convergence of $P_M^{(\Lambda)}$ to $\mu^{\otimes M}$, which is equivarent to (3.6). □

A.2 Proofs for section 4

Although the proposition 2 is proved by the completely same way as the single-population case (theorem 2 and thorem 3 in Tanabe (2001)), we present a proof for the convenience of the reader. We begin with proving a lemma that gives an estimate for a moment of $\frac{1}{\sigma_n^{i\alpha}}$.

Lemma 1. For any $\alpha > 1, T > 0, i \in I$,

$$
\begin{aligned}
&E[\frac{1}{\sigma_n^{i\alpha}}; \sigma_n^i > T] \\
&= \frac{c_i^n e^{-c_i T}}{\alpha - 1}[\frac{T^{n-\alpha}}{(n-1)!} \\
&+(I_{n-1}(T) + T I_{n-2}(T) + \frac{T^2 I_{n-3}(T)}{2!} + \cdots + \frac{T^{n-3} I_2(T)}{(n-3)!} + \frac{T^{n-2} I_1(T)}{(n-2)!}) \\
&-c_i(I_n(T) + T I_{n-1}(T) + \frac{T^2 I_{n-2}(T)}{2!} + \cdots + \frac{T^{n-2} I_2(T)}{(n-2)!} + \frac{T^{n-1} I_1(T)}{(n-1)!}),
\end{aligned}
\tag{A.8}
$$

where $I_n(T) = \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n-fold} \frac{e^{-c_i(x_1+\cdots+x_n)}}{(x_1 + \cdots + x_n + T)^{\alpha-1}} dx_1 \cdots dx_n$.

Therefore,

$$
\sum_{n=1}^{\infty} E[\frac{1}{\sigma_n^{i\alpha}}; \sigma_n^i > T] \leq \frac{c_i}{(\alpha-1)T^{\alpha-1}}. \tag{A.9}
$$

Proof of lemma 1. In the proof of this subsection we drop the suffix i for a notational convenience. Set $\tau_n = \sigma_n - \sigma_{n-1}, n \geq 1$. Since $\{\tau_n\}$ are iid with the exponential distribution of parameter c, by computation of conditional expectation conditioned on $\sigma_{n-1}, \sigma_{n-2}$, we have

$$
\begin{aligned}
&E[\frac{1}{\sigma^{\alpha_n}}; \sigma_n > T] \\
&= E[E[\frac{1}{(\tau_n + \sigma_{n-1})^\alpha} 1_{\tau_n+\sigma_{n-1}} | \sigma_{n-1}]] \\
&= \frac{1}{\alpha-1}[E[\int_0^\infty \frac{c^2 e^{-cx}}{(x+\sigma_{n-2})^{\alpha-1}} dx; \sigma_n > T] + E[\int_0^\infty \frac{c^2 e^{-c(T-\sigma_{n-2})} e^{-cx}}{(x+T)^{\alpha-1}} dx; \sigma_{n-2} \le T] \\
&-E[\int_0^\infty \frac{c^2 e^{-cx}}{(x+\sigma_{n-1})^{\alpha-1}} dx; \sigma_{n-1} > T] + E[\int_0^\infty \frac{c e^{-c(T-\sigma_{n-1})}}{T^{\alpha-1}} dx; \sigma_{n-1} \le T] \\
&-E[\int_0^\infty \frac{c^2 e^{-c(T-\sigma_{n-1})} e^{-cx}}{(x+T)^{\alpha-1}} dx; \sigma_{n-1} \le T]. \qquad \text{(A.10)}
\end{aligned}
$$

Moreover, by a similar computation we obtain

$$
E[e^{-c(T-\sigma_n)}; \sigma_n \le T] = \frac{c^n T^n}{n!} e^{-cT}, \qquad \text{(A.11)}
$$

$$
\begin{aligned}
&E[\int_0^\infty \frac{e^{-cx}}{(x+\sigma_{n-2})^{\alpha-1}} dx; \sigma_n > T] \\
&= c^n e^{-cT}(I_{n+1}(T) + T I_n(T) + \frac{T^2 I_{n-1}(T)}{2!} + \cdots + \frac{T^{n-1} I_2(T)}{(n-1)!}. \qquad \text{(A.12)}
\end{aligned}
$$

Hence we have (A.8) by substituting (A.11) and (A.12) for (A.10), and (A.9) immediately follows from (A.8). □

Proof of proposition 2(i). From the assumption, for any $\varepsilon > 0$ there exists $T > 0, a > 0$ such that $v_k(t) \le a t^{-\alpha} < \varepsilon, t > T$. Then by (3.3), for $\sigma_{N+1} > T$ we have

$$
P_{X(\sigma_N),k}(v(\sigma_{N+1})) \le \begin{cases} \frac{C_{r1}C_{p1}}{c} v_k(\sigma_{N+1}) \le \frac{C_{r1}C_{p1}}{c} \frac{a}{\sigma_{N+1}^\alpha}, X(\sigma_N) \ne k \\ 1 - \frac{C_{r2}C_{p2}}{c}(1 - v_k(\sigma_{N+1})) < 1 - \frac{C_{r2}C_{p2}}{c}(1-\varepsilon), X(\sigma_N) = k \end{cases}. \qquad \text{(A.13)}
$$

Set $\mathcal{F}_t^{X,N} = \sigma(X(s), N(s), s \le t)$. Then, by the strong Markov property of X

w.r.t. $\{\mathcal{F}_t^{X,N}\}$ [9] and (A.13),

$$
\begin{aligned}
&P(X(\sigma_{N+1}) = k, \sigma_N > T) \\
&= E[E[1_{(X(\sigma_{N+1})=k)}|\mathcal{F}_{\sigma_N}]; \sigma_N > T] = E[P_{X(\sigma_N),k}(v(\sigma_N)); \sigma_N > T] \\
&\leq E[\frac{C_{r1}C_{p1}}{c}\frac{a}{\sigma_{N+1}^{\alpha}} \cdot 1_{S\setminus\{k\}}(X(\sigma_N)) + (1 - \frac{C_{r2}C_{p2}}{c}(1-\varepsilon)) \cdot 1_k(X(\sigma_N)); \sigma_N > T] \\
&\leq E[\frac{C_{r1}C_{p1}}{c}\frac{a}{\sigma_{N+1}^{\alpha}} + (1 - \frac{C_{r2}C_{p2}}{c}(1-\varepsilon)) \cdot 1_k(X(\sigma_N)); \sigma_N > T] \\
&\leq \frac{C_{r1}C_{p1}}{c}E[\frac{a}{\sigma_{N+1}^{\alpha}}; \sigma_{N+1} > T] + (1 - \frac{C_{r2}C_{p2}}{c}(1-\varepsilon))P(X(\sigma_N) = k). \quad \text{(A.14)}
\end{aligned}
$$

Noting $P(X(\sigma_{N+1}) = k, \sigma_N \leq T) \leq P(\sigma_N \leq T) = \sum_{n=N}^{\infty}\frac{(cT)^n e^{-cT}}{n!}$,
from (A.14) we have

$$
\begin{aligned}
&P(X(\sigma_{N+1}) = k) \\
&= P(X(\sigma_{N+1}) = k, \sigma_N > T) + P(X(\sigma_{N+1}) = k, \sigma_N \leq T) \\
&\leq \frac{C_{r1}C_{p1}}{c}E[\frac{a}{\sigma_{N+1}^{\alpha}}; \sigma_{N+1} > T] + (1 - \frac{C_{r2}C_{p2}}{c}(1-\varepsilon))P(X(\sigma_N) = k) \\
&+ \sum_{n=N}^{\infty}\frac{(cT)^n e^{-cT}}{n!}.
\end{aligned}
$$

Hence, by use of lemma 1, we have

$$
\sum_{N=1}^{\infty} P(X(\sigma_N) = k) \leq \frac{c}{C_{r2}C_{p2}(1-\varepsilon)}(1 + \frac{aC_{r1}C_{p1}}{(\alpha-1)T^{\alpha-1}} + cT) < +\infty.
$$

Then, by Borel-Cantelli's lemma, we conclude $P(\bigcap_{N\geq 1}\bigcup_{n\geq N}(X(\sigma_n) = k) = 0$, which implies $P(\lim_{t\to\infty} 1_{k_i}(X^i(t)) = 0) = 1$. □

Proof of proposition 2(ii). Since

$$
P(\lim_{t\to\infty} 1_k(X(t)) = 0) = P(\cap_{N\geq 1}\cup_{n\geq N}(X(\sigma_n) = k)) = \lim_{N\to\infty} P(\cap_{n\geq N}(1_k(X(\sigma_n)) = 0)),
$$

it is sufficient to show $P(\cap_{n\geq N}(1_k(X(\sigma_n)) = 0) = 0, N \geq 1$ for the proof.

[9] Because X is a right-continuous and stochastically continuous Markov process w.r.t. $\{\mathcal{F}_t^{X,N}\}$ (see theorem 7 and the following remark 1 in section 1.4 of Gihman and Skorohod(1975)).

First we claim

$$P(\cap_{n=N}^{M}(X(\sigma_n)) \in S \setminus \{k\})) \leq E[1_{S\setminus\{k\}}(X(\sigma_n)) \prod_{n=N+1}^{M}(1-\frac{1}{c}C_{r2}C_{p2}v_k(\sigma_n))], M > N. \tag{A.15}$$

For any bounded measurable function $f^{(n)}$ on $S^{M-N-n+1} \times R_+^n$, it holds by the way to construct X that

$$\begin{aligned}
&E[f^{(n)}(X(\sigma_N), \cdots, X(\sigma_{M-n}), \sigma_{M-n+1}, \cdots, \sigma_M)| \\
&X(\sigma_N)), \cdots, X(\sigma_{M-n-1}), \sigma_N, \cdots, \sigma_M] \\
&= \sum_{l\in S} P_{X(\sigma_{M-n-1}),l}(v(\sigma_{M-n})) \\
&\cdot f^{(n)}(X(\sigma_N), \cdots, X(\sigma_{M-n-1}), l, \sigma_{M-n+1}, \cdots, \sigma_M), 1 \leq n \leq M-N-1.
\end{aligned} \tag{A.16}$$

Then, noting $P_{lk}(v) \geq \frac{1}{c}C_{r2}C_{p2}v_k, l \neq k$ by the assumption and using (A.16) repeatedly we have (A.15).

By the assumption, $v_k(t) \geq ag(t), t \geq T$ for some $a > 0$ and sufficintly large $T > 0$. Further, by applying the law of large numbers to $\{\sigma_n - \sigma_{n-1}\}$, there exists $N_0(\omega) > N$ such that $|\frac{\sigma_n(\omega)}{n} - \frac{1}{c}| < \rho, n \geq N_0(\omega)|$ and $(\frac{1}{c} - \rho)N_0(\omega) > T$ for $\frac{1}{c} > \rho > 0$ for almost all $\omega \in \Omega$.

Then for a.a. $\omega \in \Omega$, $(\frac{1}{c} + \rho)n > \sigma_n(\omega) \geq \frac{1}{c} - \rho)N_0(\omega) > T$ and $v_k(\sigma_n) \geq ag(\sigma_n) \geq ag((\frac{1}{c} + \rho)n), n \geq N_0(\omega)$.

Therefore, since $\sum_{n=N_0}^{\infty} v_k(\sigma_n) \geq \sum_{n=N_0}^{\infty} ag((\frac{1}{c}+\rho)n) = +\infty$ by the assumption w.r.t. g, $\prod_{n=N_0}^{M}(1 - \frac{1}{c}C_{r2}C_{p2}v_k(\sigma_n)) \searrow 0$ as $M \nearrow \infty$ for a.a. $\omega \in \Omega$. This implies the integrant of the right-hand of (A.15) goes to zero a.s., and we have the conclusion by the bounded convergence theorem. □

Proof of corollary 1. It is shown in the proof of theorem 1 in Samuelson and Zhang (1992) that for any pure strategy k that is iteratively strictly dominated by another pure strategy $\in S$ there exist $\theta, \rho > 0$ and time $T > 0$ such that

$u_k(t) < \theta e^{-\rho t}, t > T$ holds for any payoff monotonic dynamics. So the conculusion immediately follows by proposition 2(ii). □

Proof of proposition 3(i). First we claim that $u(t) \to u^*$ imples that $G_l^j(u^*) \leq 0, l \in S^j$ and in particular $G_l^j(u^*) = 0$ if $l \in C(u^{*i})$ for $j \in I$.

Suppose $G_h^j(u^*) > 0$ for some $h \in S^j$. Then by the continuity of G_h^j, there exists $\varepsilon > 0$ and $T > 0$ such that $\frac{\dot{u}_h^j}{u_h^j}(t) = G_h^j(u(t)) > \varepsilon, t \geq T$. From this we have $u_h^j(t) \nearrow \infty$ and get to a contradiction. Further $G_l^j(u^*) = 0$ for $l \in C(u^{*j})$ immediately follows since $G_l^j(u^*) < 0$ implies $u_l^{*j} = 0$.

Next, we have $G_l^i(u^*) < G_k^i(u^*) = 0, l \neq k \in S^i$ since k is a unique best reply and G^i is payoff monotonic. This shows u_l^i exponentially decrease to zero for $l \neq k \in S^i$. So we have $P(\lim_{t\to\infty} X^i(t) = k) = 1$ by proposition 2(i). □

Proof of proposition 3(ii). By the Lipschitz continuity of G_h^i,

$$G_h^i(u) \geq G_h^i(u^*) - C_h^i(\|u^i - u^{i*}\| + \|u^{-i} - u^{-i*}\|), \tag{A.17}$$

where $\|u\| = \sum_{l,j} |u_l^j|$ and C_h^i is a Lipschitz constant.

Next, denote by B the set of all best replies in S^i except k against u^{*-i}. Then $h_l^j \in B$ and $G_l^i(u^*) = G_k^i(u^*) = 0$ for $l \in B$ and $G_l^i(u^*) < 0$ for $l \in B^c \setminus \{k\}$ because of the payoff monotonicity of G^i.

Hence, by the continuity of G^i, there exists $\rho > 0$ and $T_1 > 0$ for some $h_1 \in B$ such that

$$(\dot{\overline{\frac{u_l^i}{u_{h_1}^i}}})(t) = (G_l^i - G_{h_1}^i)(\frac{u_l^i}{u_{h_1}^i})(t) \leq -\rho(\frac{u_l^i}{u_{h_1}^i})(t), t \geq T_1, l \in A^c \setminus \{k\}.$$

From this there exists $\beta_1 > 0$ such that $u_l^i(t) \leq \beta_1 u_{h_1}^i(t), t \geq T_1, l \in A^c \setminus \{k\}$.

Further, from the assumption that $\lim_{u\to u^*} \frac{G_l^j(u)u_l^j}{G_{h_l^j}^i(u)u_{h_l^j}^i}$ is finite, we have that there exists $\beta_2 > 0$ and $T_2 > 0$ such that $|u_l^j(t) - u_l^{*j}| < \beta_2 u_{h_l^j}^i(t), t \geq T_2, l \in S^j, j \neq i \in I$.

Thus, from (A.17) we have

$$
\begin{aligned}
G_h^i(u) &\geq -C_h^i(2\sum_{l\in B} u_l^i + 2\sum_{l\neq k\in B^c} u_l^i + \sum_{l,j\neq i} |u_l^j(t) - u_l^{*j}|) \\
&\geq -C_h^i(2\sum_{l\in B} u_l^i + 2\beta_1(m^i - |B| - 1)u_{h_1}^i + \beta_2 \sum_{l,j\neq i} u_{h_l^j}^i) \\
&> -C_1 C_h^i \sum_{l\in B} u_l^i, t \geq T = T_1 \vee T_2, h \in A,
\end{aligned}
$$

where $C_1 = 2 + 2\beta_1(m^i - |B| - 1) + \beta_2(m - m^i)$.

So we have

$$
\begin{aligned}
\sum_{l\in B} \dot{u}_l^i &> -(\sum_{l\in A} C_1 C_l^i u_l^i)(\sum_{l\in B} u_l^i) \\
&\geq -C_2(\sum_{l\in B} u_l^i)^2, t \geq T,
\end{aligned}
$$

where $C_2 = C_1 \max\{C_l^i : l \in B\}$.

Therefore, there exists $C > 0$ such that

$$
\sum_{l\in B} u_l^i(t) \geq \frac{\sum_{l\in B} u_l^i(T)}{1 + C(t-T)\sum_{l\in B} u_l^i(T)}, t \geq T.
$$

But this implies that $\underline{\lim}_{t\to\infty} t \cdot u_{h_2}^i(t) \geq \frac{1}{C|B|}$ for some $h_2 \in B$, and we conclude $P(\lim_{t\to\infty} 1_{h_2}(X^i(t)) = 0) = 0$ by proposition 2(ii). □

A.3 Proofs for section 5

The proof of the proposition is carried out based on the following lemmas. Lemma 2 extends the ergodic theorem for a homogenuous discrete time Markov chain to a homogenuous continuous time Markov chain. Lemma 3, together with lemma 4, shows the time average of a Mckean process is estimated by that of a homogenuous continuous time Markov chain by a coupling method (for details, see Lindvall (1992)).

In the proof it is a key that $r_h^i(u^*) = r_l^i(u^*), l, h \in S^i$, which assures the imitation scheme of all individuals are common at the limit state, combined with $p_{lh}^i =$

$p_h^i, l, h \in S^i, i \in I$.

Lemma 2. Let $Y^i(n), i \in I$ be a collection of a mutually independent irreducible Markov chain with state space S^i and transition probability $P = (P^i_{lh})$. Set $Z^i(t) = Y^i(N^i(t)), i \in I$ and $N(t) = \sum_{i \in I} N^i(t)$, denoting by σ_n the n -th jump time of $N(t)$ with $\sigma_0 = 0$. If $Y^i(n)$ is independent of the Poisson processes $N^i(t), i \in I$ and $P^i_{hh} > 0, h \in S^i$ for every $i \in I$, then $Y^i(n)$ and $Z^i(t)$ have the same, unique, and stationary distribution ν^i, and for any initial distribution $\bar{u} \in \Theta$,

$$P(\lim_{N\to\infty} \frac{1}{N} \int_0^{\sigma_N} 1_H(X(s))ds = \frac{\prod \nu^i_{h_i}}{\sum c^i})$$
$$= P(\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \tau_n 1_H(X(\sigma_{n-1}))ds = \frac{\prod \nu^i_{h_i}}{\sum c^i}) = 1, H = (h^1, \dots, h^n) \in S,$$

where $1_H = 1_{h^1} \otimes \cdots \otimes 1_{h^n}$ and $\tau_n = \sigma_n - \sigma_{n-1}$.

Proof of lemma 2. Since $N^i(t), i \in I$ are mutually independent Poisson processes, $N(t)$ is also a Poisson process with intensity $c = \sum c^i$. Then $\tau_n, n \geq 1$ are mutually independent and independent of $Y^i(n), i \in I$ such that $P(\tau_n > t) = e^{-ct}$.

Note the irreduciblity of Y^i and $P^i_{hh} > 0, h \in S^i$ assure that for $h, l \in S^i$ there exists an integer $n_0(h, l) > 0$ such that $P^{in}_{lh} > 0, n \geq n_0$. Then by the mutual independence of $Y^i, i \in I$, $Y(m) = (Y^1(m), \cdots, Y^n(m))$ is also an irreducible Markov chain with finite state space S. So for $i \in I$, Y^i has a unique stationary distribution $\nu^i \in int(\triangle^i)$ such that $(\nu^1, \cdots, \nu^n)$ is a unique stationary distribution of Y and

$$P(\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} 1_H(Y(n)) = \prod \nu^i_{h^i}) = 1, H \in S \tag{A.18}$$

for any initial distibution $\bar{u} \in \Theta$ (the second chapter of Hoel et al(1971)).

Then ν^i is a unique probability distribution that satisfies $\sum_{l \in S} \nu^i_l p_{lh} = \nu^i_h, h \in S^i$, which is equivarent to $< \nu^i, cP1_h >= 0, h \in S^i$. From this we find that ν^i is also a unique stationary distribution of Z^i, since Z^i is a solution of MP for $(cP^i, \bar{u^i})$.

As $\sigma(\tau_n, n \geq 1)$ and $\sigma(Y(n), n \geq 0)$ are mutually independent, it holds that

$$
\begin{aligned}
&E[\tau_m 1_H(Y(m-1))|\sigma(Y(m), m \geq 0)] = \frac{1}{c} 1_H(Y(m-1)) \\
&E[\{(\tau_m - \frac{1}{c}) 1_H(Y(m-1))\}^2 |\sigma(Y(m), m \geq 0)] \\
&= \frac{1}{c^2} 1_H(Y(m-1)) \leq \frac{1}{c^2} < +\infty.
\end{aligned}
$$

Then, by applying the law of large numbers to $\tau_m 1_H(Y(m-1))$ w.r.t. $P(\cdot|\sigma(Y(m), m \geq 0))$,

$$
\begin{aligned}
&P(\lim_{N\to\infty} \frac{1}{N}\{\sum_{m=1}^{N} \tau_m 1_H(Z(\sigma_{m-1})) - \sum_{m=1}^{N} \frac{1}{c} 1_H(Y(m-1))\} = 0|\sigma(Y(m), n \geq 0)) \\
&= P(\lim_{N\to\infty} \frac{1}{N}\{\sum_{m=1}^{N} \tau_m 1_H(Y(m-1)) - \sum_{m=1}^{N} \frac{1}{c} 1_H(Y(m-1))\} = 0|\sigma(Y(m), n \geq 0)) \\
&= 1.
\end{aligned}
$$

Hence, combined with (A.18), we conclude

$$
\begin{aligned}
&P(\lim_{N\to\infty} \frac{1}{N} \int_0^{\sigma_N} 1_H(Z(t))dt = \frac{\prod \nu^i_{h^i}}{c}) \\
&= P(\lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} \tau_m 1_H(Z(\sigma_{m-1})) = \lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} \frac{1}{c} 1_H(Y(m-1))) \\
&= E[P(\lim_{N\to\infty} \frac{1}{N}\{\sum_{m=1}^{N} \tau_m 1_H(Y(m-1)) \\
&- \sum_{m=1}^{N} \frac{1}{c} 1_H(Y(m-1))\} = 0|\sigma(Y(m), m \geq 0))] = 1.
\end{aligned}
$$

□

Under the assumption of proposition 4, for the limit state u^* of $u(t)$, denote by P^{i*} transition probability $P^i(u^*)$ for $i \in I$, that is,

$$
P^{i*}_{lh} = \begin{cases} \dfrac{r^{i*} p^i_h(u^*)}{c^i}, & h \neq l \\ 1 - \sum_{l \neq h} \dfrac{r^{i*} p^i_l(u^*)}{c^i}, & h = l \end{cases}.
$$

Set $C^i(u^*) = \{l \in S^i : p^i_l(u^*) > 0\}$ and denote by $|C^i(u^*)|$ the cardinality of $C^i(u^*)$. Further, in cases of $|C^i(u^*)| > 1$, for $0 < \varepsilon < \min\{P^{i*}_{lh}, l \neq h, h \in C^i(u^*)\} \wedge \min\{\frac{P^{i*}_{hh}}{m^i - 1}\}$ and any fixed $k^i \in C^i(u^*)$, set transition probablity $P^{i\varepsilon+}$ and $P^{i\varepsilon-}$ on S^i, $i \in I$ by

$$P^{i\varepsilon+}_{lh} = \begin{cases} P^{i*}_{lh} - \varepsilon, & h \neq k^i \in C^i(u^*) \\ P^{i*}_{lk} + (|C^i(u^*)| - 1)\varepsilon, & h = k^i \\ P^{i*}_{lh}, & h \in C^i(u^*)^c \end{cases},$$

$$P^{i\varepsilon-}_{lh} = \begin{cases} P^{i*}_{lh} + \varepsilon, & h \neq k^i \\ P^{i*}_{lh} - (m^i - 1)\varepsilon, & h = k^i. \end{cases}$$

We note that $P^{i\varepsilon-}$ has a unique stationary state since it is positive. Moreover, it is easily shown that $P^{i\varepsilon+}$ and P^{i*} also have unique stationary states that are probabilities on $C^i(u^*)$.

Let $Y^{i\varepsilon+}(n)(Y^{i-}(n)), i \in I$ be a collection of a mutually independent Markov chain on S^i that is independent of $\{N^i(t), i \in I\}$ and determined by $P^{i\varepsilon+}$(resp.$P^{i\varepsilon-}$). Set $X^{i\varepsilon+}(t) = Y^{i\varepsilon+}(N^i(t))$ ($X^{i\varepsilon-}(t) = Y^{i\varepsilon-}(N^i(t))$) and $X^{\varepsilon+}(t) = (X^{1\varepsilon+}(t), \cdots, X^{n\varepsilon+}(t))$ (resp.$X^{\varepsilon-}(t) = (X^{1\varepsilon-}(t), \cdots, X^{n\varepsilon-}(t))$).

Then , $X^{\varepsilon+}(X^{\varepsilon-})$ has a unique stationary distribution ν^{i+} (resp. ν^{i-}), and for any initial distribution $\bar{u} \in \Theta$ and $h^i \in C^i, i \in I$,

$$P(\lim_{N\to\infty} \frac{1}{N} \int_0^{\sigma_N} 1_H(X^{\varepsilon+}(t))dt = \frac{\prod \nu^{i\varepsilon+}_{h^i}}{c}) = 1 \tag{A.19a}$$

$$P(\lim_{N\to\infty} \frac{1}{N} \int_0^{\sigma_N} 1_H(X^{\varepsilon-}(t))dt = \frac{\prod \nu^{i\varepsilon-}_{h^i}}{c}) = 1 \tag{A.19b}$$

(A.19b) immediately follows from lemma 2, but for (A.19a) we need a slight modification of lemma 2.

Since $X^{i\varepsilon+}$ remains in C^i once it enters C^i, set $\sigma'_0 = \inf\{t \geq 0 : X^{i\varepsilon+}(t) \in C^i, i \in I\}$. Then $P(\sigma'_0 < +\infty) = 1$ since for $h \in C^i$,

$$\begin{aligned} &P(X^{i\varepsilon+}(t) = h, 0 \leq t \leq T) \\ &= \sum_{n=0}^{\infty} P(X^{\varepsilon+}(t) = h, 0 \leq t \leq T, \sigma_n \leq T < \sigma_{n+1}) \\ &= \sum_{n=0}^{\infty} (1 - \frac{r^{i*}}{c})^n \cdot \frac{(cT)^n}{n!} e^{-cT} = e^{-r^{i*}T} \searrow 0. \end{aligned}$$

After entering $\otimes_{i=1}^{n} C^i$, $X^{\varepsilon+}$ behaves as an irreducible Markov chain on $\otimes_{i=1}^{n} C^i$. Denote by $\mathcal{F}_{\sigma'_0}^{N,X^{\varepsilon+}}$ σ-field generated by $X^{\varepsilon+}$ and N stopped by σ'_0, i.e., $\mathcal{F}_{\sigma'_0}^{N,X^{\varepsilon+}} = \sigma\{X^{\varepsilon+}(t \wedge \sigma'_0), N(t \wedge \sigma'_0)), t \geq 0\}$. Then, by the strong Markov property of N and $X^{\varepsilon+}$ and lemma 2,

$$
\begin{aligned}
& P(\lim_{N\to\infty} \frac{1}{N} \int_0^{\sigma_N} 1_h(X^{\varepsilon+}(t))dt = \frac{\pi_h^{\varepsilon+}}{c} | \mathcal{F}_{\sigma'_0}^{N,X}) \\
&= P(\lim_{N\to\infty} \frac{N - N(\sigma'_0)}{N} \frac{1}{N - N(\sigma'_0)} \int_{\sigma'_0}^{\sigma_N} 1_h(X^{\varepsilon+}(t))dt = \frac{\pi_h^{\varepsilon+}}{c} | \mathcal{F}_{\sigma'_0}^{N,X}) \\
&= P(\lim_{N\to\infty} \frac{1}{N - N(\sigma'_0)} \int_{\sigma'_0}^{\sigma_N} 1_h(X^{\varepsilon+}(t))dt = \frac{\pi_h^{\varepsilon+}}{c} | N(\sigma'_0), X^{\varepsilon+}(\sigma'_0), \sigma'_0) \\
&= 1
\end{aligned}
$$

From this (A.19a) immediately follows.

Lemma 3. Under the assumption of proposition 4 the following statements hold.

(i)If $|C(u^{*i})| > 1, i \in I$, for any $\varepsilon > 0$ with

$$
0 < \varepsilon < \varepsilon^{*+} = \min\{\frac{1}{(|C(u^{*i})| + 1) \vee (m^i + |C(u^{*i})| - 3)}(1 - \frac{r^{i*}}{c}) \wedge \min\{P_{hl}^{i*}, l \neq h\} : i \in I\}, \tag{A.20}
$$

there exists $T > 0$ such that for the Mckean process X c.t. $(\{Q(v); v \in \Theta\})$,

$$
\begin{aligned}
& P_{H,T}(\overline{\lim_{N\to\infty}} \frac{c}{N} \int_T^{\sigma'_N} 1_K(X(s))ds > \prod \nu_{k^i}^{i\varepsilon+}) \\
& \leq P_{H,T}(\lim_{N\to\infty} \frac{c}{N} \int_T^{\sigma'_N} 1_K(X^{\varepsilon+}(s))ds > \prod \nu_{k^i}^{i\varepsilon+}) = 0, H \in S.
\end{aligned}
$$

(ii)For any $\varepsilon > 0$ with

$$
0 < \varepsilon < \varepsilon^{*-} = \min\{\frac{1}{m^i \vee 3}(1 - \frac{r^{i*}}{c}) \wedge \min\{\frac{P_{hh}^{i*}}{m^i - 1}, h \in S^i\} : i \in I\}, \tag{A.21}
$$

there exists $T > 0$ such that for the Mckean process X c.t. $(\{Q(v); v \in \Theta\})$,

$$
\begin{aligned}
& P_{H,T}(\underline{\lim_{N\to\infty}} \frac{c}{N} \int_T^{\sigma'_N} 1_K(X(s))ds < \prod \nu_{k^i}^{i\varepsilon-}) \\
& \leq P_{H,T}(\lim_{N\to\infty} \frac{c}{N} \int_T^{\sigma'_N} 1_K(X^{\varepsilon-}(s))ds < \prod \nu_{k^i}^{i\varepsilon+}) = 0, H \in S.
\end{aligned}
$$

Here $P_{H,T}$ stands for the conditional probability conditioned on $X(T)$(resp.$X^{\varepsilon+}(T)$, $X^{\varepsilon-}(T)) = H \in S$, and σ'_n is the $n-$ th jump time of $N(t)$ after T.

Proof of lemma 3. For the proof we costruct a Markov process $(\hat{X}, \hat{X}^{\varepsilon+}(\hat{X}^{\varepsilon-}))$ on $S \times S$ such that $\hat{X} \overset{D}{\simeq} X, \hat{X}^{\varepsilon+} \overset{D}{\simeq} X^{\varepsilon+}(X^{\varepsilon-})$ and $1_K(\hat{X}(t)) \le 1_K(\hat{X}^{\varepsilon+}(t)), (1_K(\hat{X}(t)) \ge 1_K(resp.\ \hat{X}^{\varepsilon-}(t)),)t \ge T, a.s.$. We only prove (i) since (ii) is shown in the same way.

By the assumption of proposition 4 and the continuity of $r^i_h(u)$ in u, there exists $T > 0$ for any ε with $0 < \varepsilon < \varepsilon^*$ such that

$$\left|\frac{r^i_h(u(t))}{c^i} - \frac{r^{i*}}{c^i}\right| < \varepsilon, h \in S^i, t \ge T. \tag{A.22a}$$

$$\left|\frac{r^i_h(u(t))p^i_l(u(t))}{c^i} - \frac{r^{i*}p^i_l(u^*)}{c^i}\right| < \varepsilon, h, l \in S^i, t \ge T. \tag{A.22b}$$

Let $(\hat{X}^i, \hat{X}^{i\varepsilon+}), i \in I$ be a collection of a mutually independent Markov process on $S^i \times S^i$ following tansition law $\hat{P}^i$ given by table (A.23) below when $N^i(t)$ jumps at time t, and set $(\hat{X}, \hat{X}^{\varepsilon+}) = (\hat{X}^1, \hat{X}^{1\varepsilon+}, \ldots, \hat{X}^n, \hat{X}^{n\varepsilon+})$. The general rule to construct $\hat{P}^i$ is as follows.

First we give probability $P^i_{hl}(u(t)) \wedge P^{i\varepsilon+}_{h'l}$ to transition $(h, h') \to (l, l)$. Next we give probability $P^i_{hl}(u(t)) - P^{i\varepsilon+}_{h'l}$ to transition $(h, h') \to (l, h')$ or probability $P^{i\varepsilon+}_{h'l} - P^i_{hl}(u(t))$ to transition $(h, h') \to (h, l)$, depending on $P^i_{hl}(u(t)) > P^{i\varepsilon+}_{h'l}$ or $P^{i\varepsilon+}_{h'l} \ge P^i_{hl}(u(t))$. Finally all residual probabilities are given to $(h, h') \to (h, h')$.

By use of (A.22) it is easily cheked that the transition probability in the following table is so well defined for ε with (A.20) that $P(\hat{X}^i(\sigma^i_n) = l|\hat{X}^i(\sigma^i_n-) = h) = P^i_{hl}(u(\sigma^i_n))$ and $P(\hat{X}^{\varepsilon+}(\sigma^i_n) = l|\hat{X}^{\varepsilon+}(\sigma^i_n-) = h) = P^{i\varepsilon+}_{hl}$.

Because it holds that

$$\sum_{l' \in S^i} \hat{P}^i_{(h,l)(h',l')}(u(\sigma^i_n)) = P^i_{hh'}u(\sigma^i_n), l \in S^i,$$
$$\sum_{h' \in S^i} \hat{P}^i_{(h,l)(h',l')}(u(\sigma^i_n)) = P^i_{ll'}u(\sigma^i_n), h \in S^i.$$

$$\begin{array}{llll} & from & to & with\ transition\ law \\ for\ any\ h, & (h,k) & (k,k) & P^i_{hk}(u(t)) \\ & & (l,k) & P^i_{hl}(u(t)) - P^{i\varepsilon+}_{kl}, l \ne k \\ & & (l,l) & P^{i\varepsilon+}_{kl}, l \ne k \end{array} \tag{A.23a}$$

$$\begin{array}{llll} for\ any\ h \neq k, & (k,h) & (k,k) & P^{i\varepsilon+}_{hk} \\ & & (k,h) & P^{i}_{kk}(u(t)) - P^{i\varepsilon+}_{hk} \\ & & (h,h) & P^{i}_{kh}(u(t)) \\ & & (l,h) & P^{i}_{kl}(u(t)) - P^{i\varepsilon+}_{hl}, l \neq h,k \\ & & (l,l) & P^{i\varepsilon+}_{hl}, l \neq h,k \end{array} \tag{A.23b}$$

$$\begin{array}{llll} for\ any\ h \neq k, & (h,h) & (k,k) & P^{i}_{hk}(u(t)) \\ & & (h,k) & P^{i\varepsilon+}_{hk} - P^{i}_{hk}(u(t)) \\ & & (h,h) & 1 - P^{i\varepsilon+}_{hk} - \sum_{l\neq h,k} P^{i}_{hl}(u(t)) \\ & & (l,h) & P^{i}_{hl}(u(t)) - P^{i\varepsilon+}_{hl}, l \neq h,k \\ & & (l,l) & P^{i\varepsilon+}_{hl}, l \neq h,k \end{array} \tag{A.23c}$$

$$\begin{array}{llll} for\ any\ h \neq k, & (h,h') & (k,k) & P^{i}_{hk}(u(t)) \\ h' \neq k, h \neq h' & & (h,k) & P^{i\varepsilon+}_{h'k} - P^{i}_{hk}(u(t)) \\ & & (h,h) & P^{i\varepsilon+}_{h'h} \\ & & (h,h') & 1 - P^{i\varepsilon+}_{h'h} - P^{i\varepsilon+}_{h'k} \\ & & & - \sum_{l\neq h,k} P^{i}_{hl}(u(t)) \\ & & (h',h') & P^{i}_{hh'}(u(t)) \\ & & (l,h') & P^{i}_{hl} - P^{i\varepsilon+}_{h'l}, l \neq h,h',k \\ & & (l,l) & P^{i\varepsilon+}_{h'l}, l \neq h,h',k \end{array} \tag{A.23d}$$

By the construction of the transition law, obviously state (k,h) with $h \neq k$ in $S^i \times S^i$ never occurs when $(\hat{X}^i, \hat{X}^{i\varepsilon+})$ begins at T on the condition $\hat{X}^i(T) = \hat{X}^{i\varepsilon+}(T)$. So it holds that

$$P(\overline{\lim_{N\to\infty}} \frac{1}{N} \int_T^{\sigma'_N} 1_K(\hat{X})(t)dt \leq \overline{\lim_{N\to\infty}} \frac{1}{N} \int_T^{\sigma'_N} 1_K(\hat{X}^{\varepsilon+})(t)dt | \hat{X}(T) = \hat{X}^{\varepsilon+}(T) = H) = 1.$$

Then we have

$$\begin{aligned} & P_{H,T}(\overline{\lim_{N\to\infty}} \frac{c}{N} \int_T^{\sigma'_N} 1_K(\hat{X}(t))dt > \prod \nu^{\varepsilon+}_{k^i}) \\ & \leq P_{H,T}(\overline{\lim_{N\to\infty}} \frac{c}{N} \int_T^{\sigma'_N} 1_K(\hat{X}^{\varepsilon+}(t))dt > \prod \nu^{\varepsilon+}_{k^i}), H \in S, \end{aligned}$$

which shows the conclusion since $P_{H,T}(\overline{\lim_{N\to\infty}} \frac{c}{N} \int_T^{\sigma'_N} 1_K(\hat{X}^{\varepsilon+}(t))dt = \prod \nu^{\varepsilon+}_{k^i}) = 1$ from (A.19a). □

Lemma 4. For $u^* \in \Theta$ with $|C^i(u^*)| > 1, i \in I$, there exists a positive sequence $\varepsilon_n \downarrow 0$ such that $\nu^{i\varepsilon_n +}, \nu^{i\varepsilon_n -} \to \nu^{i*}, i \in I$, where $\nu^{i\varepsilon_n +}(\nu^{i\varepsilon_n -}, \nu^{i*})$ is a unique stationary distribution of a Markov chain of which transition probability is $P^{i\varepsilon_n +}$ (resp. $P^{i\varepsilon_n -}, P^{i*}$).

Moreover even for $u^* \in \Theta$ with $|C^j(u^*)| = 1$ for some $j \in I$, there still exists a positive sequence $\varepsilon_n \downarrow 0$ such that $\nu^{i\varepsilon_n -} \to \nu^{i*}, i \in I$.

Proof of lemma 4. There exists a uique stationary distribution $\pi^{i\varepsilon_n +}(\pi^{i\varepsilon_n -}, \pi^{i*})$ of a Markov chain of which transition probability is $P^{i\varepsilon_n +}$(resp. $P^{i\varepsilon_n -}, P^{i*}$). By compactness of $\triangle^i$, there exist a positive sequence $\varepsilon_n \downarrow 0$ and a $\nu'^i \in \triangle^i$ such that $\nu^{i\varepsilon_n +} \to \nu'^i$ for all $i \in I$. Since it holds that $\nu^{i\varepsilon_n +} P^{i\varepsilon_n +} = \nu^{i\varepsilon_n +}$, we have $\nu'^i P^{i*} = \nu'^i$ by taking the limit. Hence we have $\nu'^i = \nu^{i\varepsilon_n +}$ by the uniqueness of stationary distribution of P^{i*}.

By the same argument, we can choose a subsequence $\varepsilon'_n \downarrow 0$ such that $\nu^{\varepsilon_n -} \to \nu^{i*}$. Futher, for $\nu^{i\varepsilon_n -}$ and ν^{i*} this argument is still valid in the case of $|C^j(u^*)| = 1$ for some $j \in I$. $\square$

Proof of proposition 4. First we note that u^{*i} is the unique stationary distribution c.t.P^{i*},i.e., $\pi^{i*} = u^{*i}$ in lemma 4 since u^* is a stationary population state of (2.4).

For the case of $|C^i(u^*)| > 1, i \in I$, let ε_n be a positive sequence in lemma 4. Then by lemma 3, there exists a $T_n > 0$ such that for $K \in \otimes C^i(u^*)$

$$P_{H,T_n}(\prod \nu_{k^i}^{i\varepsilon_n -} \le \varliminf_{N\to\infty} \frac{c}{N} \int_{T_n}^{\sigma'_N} 1_K(X(s))ds \le \varlimsup_{N\to\infty} \frac{c}{N} \int_{T_n}^{\sigma'_N} 1_K(X(s))ds \le \prod \nu_{k^i}^{\varepsilon_n +})$$
$$= 1, H \in S, \tag{A.24}$$

where σ'_N is the $N-$th jump time of $N(t)$ after T_n.

Therefore,

$$P(\prod \nu_{k^i}^{\varepsilon_n -} \le \underline{\lim}_{N\to\infty} \frac{c}{N}\int_0^{\sigma_N} 1_K(X(s))ds \le \overline{\lim}_{N\to\infty} \frac{c}{N}\int_0^{\sigma_N} 1_K(X(s))ds \le \prod \nu_{k^i}^{\varepsilon_n +})$$
$$= P(\prod \nu_{k^i}^{\varepsilon_n -} \le \underline{\lim}_{N\to\infty} \frac{c}{N}\int_{T_n}^{\sigma'_N} 1_K(X(s))ds \le \overline{\lim}_{N\to\infty} \frac{c}{N}\int_{T_n}^{\sigma'_N} 1_K(X(s))ds \le \prod \nu_{k^i}^{\varepsilon_n +})$$
$$= P(\prod \nu_{k^i}^{\varepsilon_n -} \le \underline{\lim}_{N\to\infty} \frac{c}{N}\int_{T_n}^{\sigma'_N} 1_K(X(s))ds \le \overline{\lim}_{N\to\infty} \frac{c}{N}\int_{T_n}^{\sigma'_N} 1_K(X(s))ds \le \prod \nu_{k^i}^{\varepsilon_n +}$$
$$|X(T_n))$$
$$= 1. \tag{A.25}$$

Since $\nu^{i\varepsilon_n +}, \nu^{i\varepsilon_n -} \to u^{i*}$, from (A.25) we obtain

$$P(\lim_{N\to\infty} \frac{c}{N}\int_0^{\sigma_N} 1_K(X(s))ds = \prod u_{k^i}^{i*}) = 1, \tag{A.26}$$

which is equivarent to

$$P(\lim_{N\to\infty} \frac{1}{\sigma_N}\int_0^{\sigma_N} 1_K(X(s))ds = \prod u_{k^i}^{i*}) = 1$$

by $\frac{\sigma_N}{N} \to \frac{1}{c}, a.s.$.

For any $T > 0$, there exists some positive integer $N_T(\omega)$ such that $\sigma_{N_T(\omega)} \le T < \sigma_{N_T(\omega)+1}$ for almost all $\omega \in \Omega$. Hence we have

$$\frac{1}{\sigma_{N_T+1}}\int_0^{\sigma_{N_T}} 1_K(X(s))ds \le \frac{1}{T}\int_0^T 1_K(X(s))ds \le \frac{1}{\sigma_{N_T}}\int_0^{\sigma_{N_T+1}} 1_K(X(s))ds, a.s..$$

Then noting that $\frac{\sigma_N}{N} \to \frac{1}{c}$ means $\frac{\sigma_{N_T+1}}{\sigma_{N_T}} \to 1$ as $T \to \infty$, we conclude

$$P(\lim_{T\to\infty} \frac{1}{T}\int_0^T 1_K(X(s))ds = \prod u_{k^i}^{i*}) = 1, K \in \otimes C^i(u^*), i \in I.$$

For the case where $|C^j(u^*)| = 1$ for some $j \in I$, set $J = \{j \in I : |C^j(u^*)| = 1\}$. Then we notice that in (A.25) the inequality in the lefthand still holds and the inequality in the righthand can be replaced by one based on $\prod_{i\in I\setminus J} 1_{k^i}(X^i(s))$. Thus we have in place of (A.25)

$$P(\prod \nu_{k^i}^{i\varepsilon_n -} \le \underline{\lim}_{N\to\infty} \frac{c}{N}\int_{T_n}^{\sigma'_N} 1_K(X(s))ds \le \overline{\lim}_{N\to\infty} \frac{c}{N}\int_{T_n}^{\sigma'_N} 1_K(X(s))ds$$
$$\le \prod_{i\notin J} \nu_{k^i}^{i\varepsilon_n +}|X(T_n)) = 1, \tag{A.25$'$}$$

which leads to

$$P(\lim_{N\to\infty} \frac{c}{N} \int_0^{\sigma_N} 1_K(X(s))ds = \prod u_{k^i}^{i*} = 1) = 1, \tag{A.26$'$}$$

by $v_{k^i}^{i\varepsilon_n -} \to u_{k^j}^{j*} = 1$, $j \in J$. From this we have the conculusion for the case. □