Generalized Confluent Hypergeometric Systems on Grassmann Variety.

Y. Murata and N. M. J. Woodhouse

Abstract

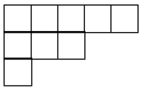
The theory of Generalized Confluent Hypergeometric Function and Generalized Confluent Hypergeometric System defined on matrix space was initiated by Gelfand and developed by H. Kimura et al., using tools related to Young tableaux and twisted cycles. The aim of this paper is to give a concrete expression of Generalized Confluent Hypergeometric System on Grassmann Variety. Results will be applied to the study of the relationship between Generalized Confluent Hypergeometric Systems and Matrix Painlevé Systems in the forthcoming paper.

1 Introduction

Generalization of hypergeometric equations with regular singularities have been studied by many authors. Especially, Aomoto [1], Gelfand [2, 3], Yoshida [16], Matumoto et al. [13] considered integral expressions of hypergeometric functions and dealt generalized hypergeometric systems with regular singularities defined on matrix space. Gelfand et al. [4] generalized this method to define confluent hypergeometric functions defined on matrix space. Inspired Gelfand's work, Kimura et al. [5, 7, 8, 9, 10, 11, 12] defined the concept of Generalized Confluent Hypergeometric Function and Generalized Confluent Hypergeometric System defined on matrix space, using tools related to Young tableaux and twisted cycles.

The aim of this paper is to give a concrete expression of Generalized Confluent Hypergeometric System on Grassmann Variety. Results will be applied to the study of Matrix Painlevé Systems in the forthcoming paper [15]. To explain our results, first we review various concepts in the theory of Generalized Confluent Hypergeometric Function according to [5, 10].

Let λ be a symbol which expresses a Young tableau of weight n. If the number of rows of λ is l and numbers of boxes of each row are $\lambda_0, \lambda_1, \cdots, \lambda_{l-1}$, we write as $\lambda = (\lambda_0, \lambda_1, \cdots, \lambda_{l-1})$. For example, Figure 1 expresses Young tableau $\lambda = (5, 3, 1)$. We denote the weight $\lambda_0 + \lambda_1 + \cdots + \lambda_{l-1}$ by $|\lambda|$.



$\lambda = (5, 3, 1), \ |\lambda| = 9$

Figure 1.

Definition 1. *For* $\lambda = (\lambda_0, \lambda_1, \cdots, \lambda_{l-1})$*, we define an abelian group* H_λ *as follows:*

$$\mathrm{H}_\lambda = \mathrm{J}_{\lambda_0} \otimes \cdots \otimes \mathrm{J}_{\lambda_{l-1}} \subset GL(n, \mathbf{C}),$$

$$\mathrm{J}_{\lambda_k} = \left\{ \sum_{0 \leq i < \lambda_k} h_i^{(k)} \Lambda^i \;\middle|\; h_i^{(k)} \in \mathbf{C}, h_0^{(k)} \neq 0 \right\},$$

where Λ *is a shift matrix of size* λ_k*:*

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

We call H_λ *the Jordan group associated with Young tableau* λ*. We call* $\mathrm{PH}_\lambda = \mathrm{H}_\lambda/\mathcal{Z}$ *the projective Jordan group associated with Young tableau* λ*, where* $\mathcal{Z}$ *is the center of* $GL(n, \mathbf{C})$*.*

For example, $\mathrm{H}_{(2,1,1)}$ is the group of all matrices of the form:

$$\begin{pmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \qquad (acd \neq 0).$$

We often use the expression

$$\begin{aligned} h &= (h^{(0)}, \cdots, h^{(l-1)}) \\ &= ([h_0^{(0)}, \cdots, h_{\lambda_0-1}^{(0)}], \cdots, [h_0^{(l-1)}, \cdots, h_{\lambda_{l-1}-1}^{(l-1)}]) \end{aligned}$$

to express an element $h \in \mathrm{H}_\lambda$.

Definition 2. *We introduce a biholomorphic mapping* ι *by*

$$\iota : \mathrm{H}_\lambda \longrightarrow \prod_{k=0}^{l-1} (\mathbf{C}^* \times \mathbf{C}^{\lambda_k - 1}),$$

$$\begin{aligned} h = ([h_0^{(0)}, \cdots, h_{\lambda_0-1}^{(0)}], \cdots, [h_0^{(l-1)}, \cdots, h_{\lambda_{l-1}-1}^{(l-1)}]) \\ \longmapsto \iota(h) = (h_0^{(0)}, \cdots, h_{\lambda_0-1}^{(0)}, \cdots, h_0^{(l-1)}, \cdots, h_{\lambda_{l-1}-1}^{(l-1)}), \end{aligned}$$

and we denote the induced biholomorphic mapping between $\widetilde{\mathrm{H}}_\lambda$ *and* $\prod_{k=0}^{l-1}(\widetilde{\mathbf{C}}^* \times \mathbf{C}^{\lambda_k-1})$ *by the same symbol* ι*, where* $\widetilde{\mathrm{H}}_\lambda$ *and* $\prod_{k=0}^{l-1}(\widetilde{\mathbf{C}}^* \times \mathbf{C}^{\lambda_k-1})$ *are the universal coverings of* H_λ *and* $\prod_{k=0}^{l-1}(\mathbf{C}^* \times \mathbf{C}^{\lambda_k-1})$*. We also use the same symbol* ι *to denote the biholomorphic mapping*

$$\iota : \mathrm{J}_{\lambda_k} \longrightarrow \mathbf{C}^* \times \mathbf{C}^{\lambda_k - 1}, [h_0^{(k)}, \cdots, h_{\lambda_k-1}^{(k)}] \longmapsto (h_0^{(k)}, \cdots, h_{\lambda_k-1}^{(k)})$$

and the induced biholomorphic mapping between $\widetilde{\mathrm{J}}_{\lambda_k}$ *and* $\widetilde{\mathbf{C}}^* \times \mathbf{C}^{\lambda_k-1}$*.*

Let χ_λ be a character of $\widetilde{\mathrm{H}}_\lambda$, then χ_λ is expressed as

$$\chi_\lambda(h) = \chi_\lambda((h^{(0)}, \cdots, h^{(l-1)})) = \chi_{\lambda_0}(h^{(0)}) \cdots \chi_{\lambda_{l-1}}(h^{(l-1)}),$$

where χ_{λ_k} is a character of $\widetilde{\mathrm{J}}_{\lambda_k}$. In order to express χ_{λ_k} explicitly, we introduce functions θ_i as follows:

Definition 3. *For the variable* $\mathbf{v} = (v_0, v_1, v_2, \cdots)$ $(v_0 \neq 0)$*, we define* $\theta_i(\mathbf{v})$ $\quad (i = 0, 1, 2, \cdots)$ *by the generating function*

$$\begin{aligned} \log(v_0 + v_1 T + v_2 T^2 + \cdots) &= \log v_0 + \log(1 + \frac{v_1}{v_0} T + \frac{v_2}{v_0} T^2 + \cdots) \\ &= \sum_{i=0}^{\infty} \theta_i(\mathbf{v}) T^i. \end{aligned}$$

By this definition, we obtain

$$\begin{aligned} \theta_0 &= \log v_0 \\ \theta_1 &= \frac{v_1}{v_0} \\ \theta_2 &= \frac{v_2}{v_0} - \frac{1}{2}\left(\frac{v_1}{v_0}\right)^2 \\ \theta_3 &= \frac{v_3}{v_0} - \frac{v_1 v_2}{(v_0)^2} + \frac{1}{3}\left(\frac{v_1}{v_0}\right)^3 \\ &\vdots \quad . \end{aligned}$$

We note that $\theta_i(\mathbf{v})$ is a rational function of $v_0, v_1, \cdots, v_i$ and $\theta_i(k\mathbf{v}) = \theta_i(\mathbf{v})$ $(i = 1, 2, \cdots)$ for any $k \in \mathbf{C}^*$.

Proposition A . *Under the above definitions,* χ_{λ_k} *is expressed as*

$$\begin{aligned} \chi_{\lambda_k}(h^{(k)}) &= \chi_{\lambda_k}([h_0^{(k)}, \cdots, h_{\lambda_k - 1}^{(k)}]) \\ &= \exp\left(\sum_{0 \le i < \lambda_k} \alpha_i^{(k)} \theta_i(\iota(h^{(k)}))\right) \\ &= (h_0^{(k)})^{\alpha_0^{(k)}} \exp\left(\sum_{1 \le i < \lambda_k} \alpha_i^{(k)} \theta_i(\iota(h^{(k)}))\right), \end{aligned}$$

where $\alpha^{(k)} = (\alpha_0^{(k)}, \cdots, \alpha_{\lambda_k - 1}^{(k)}) (\in \mathbf{C}^{\lambda_k})$ *are constants. Conversely, a function* χ_{λ_k} *defined as the above way with some constants* $\alpha^{(k)}$ *becomes a character of* $\widetilde{\mathrm{J}}_{\lambda_k}$.

Therefore, the character χ_λ is expressed by powers of $h_0^{(k)}$ $(k = 0, \cdots, l-1)$ and exponential functions with a tuple of constants $\alpha = (\alpha^{(0)}, \cdots, \alpha^{(l-1)})$. We also use the notation $\chi_\lambda(h, \alpha)$ to express the constants α explicitly.

Definition 4. *Let* $\lambda = (\lambda_0, \cdots, \lambda_{l-1})$ *be a Young tableau of weight* n*. A tableau* $\mu = (\mu_0, \cdots, \mu_{l-1})$ *of weight* $r (\leq n)$ *is called a subtableau of* λ*, if and only if* μ *satisfies the condition:*

$$0 \leq \mu_k \leq \lambda_k \quad (k = 0, \cdots, l-1).$$

In this definition, we don't suppose μ is a Young tableau. For example, if $\lambda = (2,1,1)$, then subtableau μ of weight 2 can be $(2,0,0), (1,1,0), (1,0,1), (0,1,1)$.

Suppose r and n are integers s.t. $0 < r < n$ and let $M(r,n)$ be the set of all $r \times n$ matrices with complex components. Further we set $M_0(r,n) = \{Z \in M(r,n) \mid rank Z = r\}$. $M(r,r)$ means the set of $r \times r$ matrices with complex components. Using a Young tableau $\lambda = (\lambda_0, \cdots, \lambda_{l-1})$, we express a matrix $Z (\in M(r,n))$ as $Z = [Z^0 \cdots Z^{l-1}]$, where $Z^k = [Z_0^k \cdots Z_{\lambda_k - 1}^k]$ is a $r \times \lambda_k$ matrix and Z_i^k $(i = 0, \cdots, \lambda_k - 1)$ are column vectors. For a subtableau $\mu = (\mu_0, \cdots, \mu_{l-1})$ of weight r included in λ, we set $Z_\mu = [Z_0^0 \cdots Z_{\mu_0 - 1}^0 | \cdots | Z_0^{l-1} \cdots Z_{\mu_{l-1}-1}^{l-1}] (\in M(r,r))$.

Definition 5. *For* $\lambda = (\lambda_0, \cdots, \lambda_{l-1})$ $(|\lambda| = n)$ *and an integer* r $(0 < r < n)$*, let*

$$\begin{aligned} Z_\lambda = \{Z \in M_0(r,n) | & \textit{ For any subtableau } \mu \textit{ of weight } r \textit{ included in } \lambda, \\ & \det Z_\mu \neq 0\} \subset M_0(r,n). \end{aligned}$$

We call Z_λ *the generic stratum of* $M_0(r,n)$ *with respect to* λ*. And let*

$$E_\lambda = \left\{ (\mathbf{s}, Z) \in \mathbf{P}^{r-1} \times Z_\lambda \mid \mathbf{s} Z_0^k \neq 0 \quad (k = 0, \cdots, l-1) \right\} \ \subset \mathbf{P}^{r-1} \times Z_\lambda,$$

where $\mathbf{s} = (s_0, \cdots, s_{r-1})$ *is a homogeneous coordinate in* $\mathbf{P}^{r-1}$*. We denote the natural projection by* $\phi_1 : E_\lambda \to Z_\lambda, (\mathbf{s}, Z) \mapsto Z$*.*

A fiber $E_\lambda(Z) = \phi_1^{-1}(Z)$ is a set obtained from $\mathbf{P}^{r-1}$ by subtracting l different hyperplanes.

Definition 6. *Let* $\lambda = (\lambda_0, \cdots, \lambda_{l-1})$ $(|\lambda| = n)$ *be a Young tableau and let* $\alpha = (\alpha^{(0)}, \cdots, \alpha^{(l-1)})$ $(\in \mathbf{C}^n)$ *be constants which satisfy the condition* $\alpha^{(k)} = (\alpha_0^{(k)}, \cdots, \alpha_{\lambda_k - 1}^{(k)}) \in \mathbf{C}^{\lambda_k}$ *and* $\alpha_0^{(0)} + \cdots + \alpha_0^{(l-1)} = -r$*. For* λ, α*, we consider a system*

$$G_{\lambda,\alpha} : \begin{cases} L_{km} F = \alpha_m^{(k)} F, & (0 \leq k \leq l-1, \ 0 \leq m \leq \lambda_k - 1) \\ M_{ij} F = -\delta_{ij} F, & (0 \leq i, j \leq r-1) \\ \Box_{ijpq} F = 0 & (0 \leq i, j \leq r-1, \ 0 \leq p, q \leq n-1) \end{cases}$$

defined on Z_λ*, where*

$$\begin{aligned} & L_{km} = \sum_{q=0}^{r-1} \sum_{p=A_k+m}^{A_{k+1}-1} z_{q,p-m} \frac{\partial}{\partial z_{qp}} \\ & (A_0 = 0, A_k = \lambda_0 + \cdots + \lambda_{k-1} \ (k = 1, \cdots, l)) \\ & M_{ij} = \sum_{p=0}^{n-1} z_{ip} \frac{\partial}{\partial z_{jp}} \\ & \Box_{ijpq} = \frac{\partial^2}{\partial z_{ip} \partial z_{jq}} - \frac{\partial^2}{\partial z_{iq} \partial z_{jp}} \\ & \delta_{ij} \textit{ is Kronecker's } \delta. \end{aligned}$$

We call this system the Generalized Confluent Hypergeometric System (GCHS) on Z_λ.

Let $\mathcal{O}(Z_\lambda) = \{$ analytic functions defined on $Z_\lambda\}$. We consider two properties for functions in $\mathcal{O}(Z_\lambda)$:

(A) $F(Z)(\in \mathcal{O}(Z_\lambda))$ satisfies $F(KZ) = h(K)F(Z)$ for any $K \in GL(r)$ where $h(K) = (detK)^{-1}$.

(B) $F(Z)(\in \mathcal{O}(Z_\lambda))$ satisfies $F(ZL) = F(Z)\chi_\lambda(L, \alpha)$ for any $L \in PH_\lambda$.

Definition 7. *We define three subsets in $\mathcal{O}(Z_\lambda)$ as follows:*

$$
\begin{aligned}
&S_A = \{F \in \mathcal{O}(Z_\lambda) \mid F \textit{ has the property } (A)\} \\
&S_{A,B} = \{F \in \mathcal{O}(Z_\lambda) \mid F \textit{ has the properties } (A) \textit{ and } (B)\} \\
&S = \{F \in \mathcal{O}(Z_\lambda) \mid F \textit{ is a solution of GCHS } G_{\lambda,\alpha}\}.
\end{aligned}
$$

We note that $S_{A,B} \subset S_A \subset \mathcal{O}(Z_\lambda)$. Kimura et al. [5] showed the following facts:

Proposition B . *It holds that $S \subset S_{A,B}$.*

Proposition C . *$G_{\lambda,\alpha}$ is holonomic.*

The set $M_0(r, n)$ is naturally acted by $GL(r)$ from left-hand, and acted by H_λ from right-hand. $GL(r)\backslash M_0(r, n)$ is the Grassmann variety $Gr(r, n)$. And $U_\lambda = GL(r)\backslash Z_\lambda$ is an open set of $Gr(r, n)$. Since Z_λ is H_λ - invariant, $D_\lambda = U_\lambda/\mathrm{H}_\lambda = GL(r)\backslash Z_\lambda/\mathrm{H}_\lambda$ $(\subset Gr(r, n)/\mathrm{H}_\lambda)$ is an open manifold of the variety $Gr(r, n)/\mathrm{H}_\lambda$. Further we note that $D_\lambda = U_\lambda/\mathrm{PH}_\lambda \subset G_r(r, n)/\mathrm{PH}_\lambda$. From these facts and the property

$$F(KZL) = h(K)F(Z)\chi_\lambda(L, \alpha) \qquad K \in GL(r),\ L \in \mathrm{PH}_\lambda$$

for a solution $F(Z)$ of $G_{\lambda,\alpha}$, we find that any solution $F(Z)$ of $G_{\lambda,\alpha}$ is expressible by a certain analytic function defined on D_λ. Relations of sets E_λ, Z_λ, U_λ and D_λ are as follows:

$$\begin{tikzcd}
E_\lambda \subset \mathbf{P}^{r-1} \times Z_\lambda \arrow[d, "\phi_1"] \\
Z_\lambda \subset M_0(r, n) \arrow[d, "\phi_2"] \\
U_\lambda = GL(r)\backslash Z_\lambda \subset Gr(r, n) \arrow[d, "\pi"] \\
D_\lambda = U_\lambda/\mathrm{PH}_\lambda \subset Gr(r, n)/\mathrm{PH}_\lambda
\end{tikzcd}$$

Here ϕ_1, ϕ_2, π are natural projections.

We can construct a solution of $G_{\lambda,\alpha}$ by integrating a certain differential form on E_λ. Let $\alpha = (\alpha^{(0)}, \cdots, \alpha^{(l-1)})$ $(\in \mathbf{C}^n)$ be constants which satisfy the condition $\alpha^{(k)} = (\alpha_0^{(k)}, \cdots, \alpha_{\lambda_k-1}^{(k)}) \in \mathbf{C}^{\lambda_k}$ and $\alpha_0^{(0)} + \cdots + \alpha_0^{(l-1)} = -r$. We define a $(r-1)$-form $\omega(\mathbf{s}, Z, \alpha)$ on E_λ as follows:

Definition 8.

$$\begin{aligned} \omega(\mathbf{s}, Z, \alpha) &= \chi_\lambda(\iota^{-1}(\mathbf{s}Z), \alpha)\sigma \\ &= \prod_{k=0}^{l-1} (\mathbf{s}Z_0^k)^{\alpha_0^{(k)}} \exp\left(\sum_{1 \le i < \lambda_k} \alpha_i^{(k)} \theta_i(\mathbf{s}Z_0^{(k)}, \cdots, \mathbf{s}Z_{\lambda_k-1}^{(k)}) \right) \cdot \sigma \end{aligned}$$

where

$$\begin{aligned} &(\mathbf{s}, Z) \in E_\lambda \subset \mathbf{P}^{r-1} \times Z_\lambda \\ &\sigma = \sum_{0 \le k \le r-1} (-1)^k s_k ds_0 \wedge \cdots \wedge ds_{k-1} \wedge ds_{k+1} \wedge \cdots \wedge ds_{r-1}. \end{aligned}$$

From the property of θ_i and the assumption for α, we find that $\omega(k\mathbf{s}, Z, \alpha) = \omega(\mathbf{s}, Z, \alpha)$ $(k \in \mathbf{C}^*)$. So $\omega(\mathbf{s}, Z, \alpha)$ is an analytic $(r-1)$-form on E_λ.

Definition 9. *Using a twisted cycle $\Delta(Z)$ on E_λ, we define the Generalized Confluent Hypergeometric Function (GCHF) of type λ as*

$$F(Z, \alpha) = \int_{\Delta(Z)} \omega(\mathbf{s}, Z, \alpha).$$

$F(Z, \alpha)$ is an analytic function defined on Z_λ $(\subset M_0(r, n))$.

Proposition D . *$F(Z, \alpha)$ is a solution of $G_{\lambda,\alpha}$.*

In section 2, we show the concrete expressions of GCHS on $U_\lambda \subset Gr(r, n)$. First we give preparatory propositions. Using these results, we show the expression of GCHS on U_λ (Theorem 1). Further we obtain another expression of GCHS on some manifold (Thoerem 2). Relating to Theorem 2, we give a conjecture on the expression of GCHS on D_λ. Proofs of propositions and theorems are given in section 3. Results for the case $|\lambda| = 4$ are studied and essentially applied to the study of Matrix Painlevé Systems in the forthcoming paper [15]. Matrix Painlevé Systems are defined on D_λ and they are derived from a Anti-self-Dual Yang-Mills equation defined on U_λ (See [14]).

2 Main results

In order to treat the expression of GCHS on $U_\lambda \subset Gr(r, n)$, we first consider various manifolds related to GCHF and GCHS.

Let $F_{1,r}$ be a flag manifold:

$$\begin{aligned} F_{1,r} = \{(l, S_r) |\ & l \text{ is a 1-dim linear subspace in } \mathbf{C}^n, \\ & S_r \text{ is a } r\text{-dim linear subspace in } \mathbf{C}^n \text{ s.t. } l \subset S_r\}. \end{aligned}$$

Then we have a double fibration:

$$\begin{tikzcd}
& F_{1,r} \arrow[dl, "\psi"'] \arrow[dr, "\phi"] & \\
\mathbf{P}^{n-1} & & Gr(r,n)
\end{tikzcd}$$

Here we set

$$\begin{aligned}
&U = \{Z \in Gr(r,n) |\ \text{Equivalent class } Z \text{ has a representative } [I\,W], \\
&\qquad\qquad W \in M(r, n-r)\} \simeq \mathbf{C}^{r\times(n-r)}, \\
&F_U = \phi^{-1}(U) \subset F_{1,r}, \\
&V = \psi \cdot \phi^{-1}(U) = \psi(F_U).
\end{aligned}$$

and set

$$\tilde{Z} = \{Z = [Z^0 \cdots Z^{r-1} Z^r \cdots Z^{n-1}] \in M_0(r,n) \mid det[Z^0 \cdots Z^{r-1}] \neq 0\},$$

where Z^i are column vectors of Z. Then we obtain the following Lemmas:

Lemma 1. *It holds that*

(1) F_U *is biholomorphic to* $\mathbf{P}^{r-1} \times U$.

(2) $V = \mathbf{P}^{n-1} - H$ *where* $H = \{(\underbrace{0, \cdots, 0}_{r} \underbrace{*, \cdots, *}_{n-r})\}$.

Lemma 2. $\mathbf{P}^{r-1} \times \tilde{Z} \supset \mathbf{P}^{r-1} \times Z_\lambda \supset E_\lambda$.

Further we note that $U = GL(r)\backslash\tilde{Z} \supset GL(r)\backslash Z_\lambda = U_\lambda$. So we have the following diagram:

$$\begin{tikzcd}
& \mathbf{P}^{r-1} \times \tilde{Z} \supset \mathbf{P}^{r-1} \times Z_\lambda \supset E_\lambda \arrow[d, "\phi_0"] \arrow[drr, "\phi_1"] & & \\
& F_{1,r} \supset F_U \simeq \mathbf{P}^{r-1} \times U \supset \mathbf{P}^{r-1} \times U_\lambda \arrow[dl, "\psi"'] \arrow[dr, "\phi"] & & Z_\lambda \subset M_0(r,n) \arrow[dl, "\phi_2"] \\
V & & U \supset U_\lambda = GL(r)\backslash Z_\lambda \arrow[d, "\pi"] & \\
& & D_\lambda = U_\lambda/\mathrm{PH}_\lambda &
\end{tikzcd}$$

The integrand $\omega(\mathbf{s}, Z, \alpha)$ of GCHF $F(Z, \alpha)$ is defined on the manifold E_λ, and GCHF $F(Z, \alpha)$ itself is a function on the manifold Z_λ. GCHS $G_{\lambda,\alpha}$ is a system defined on Z_λ. The theory of GCHF and GCHS has been developed on the manifold on E_λ and on Z_λ to keep the symmetry in variables, but if we want to apply this theory to other systems, we need to consider GCHF and GCHS defined on the manifold U_λ and on the manifold D_λ.

In order to state main theorems, we introduce expressions of variables in Z_λ, U_λ and D_λ. We express variables as:

$$\begin{array}{l} Z_\lambda \ni Z = [V\ V^{'}] = U[I\ W] \\ U_\lambda \ni W \simeq [I\ W] = A[I\ T]B \\ D_\lambda \ni \mathbf{t} \simeq [I\ T], \end{array}$$

where

$$V = \begin{bmatrix} z_{0\ 0} & \cdots & z_{0\ r-1} \\ \vdots & & \vdots \\ z_{r-1\ 0} & \cdots & z_{r-1\ r-1} \end{bmatrix} \in GL(r), \quad V^{'} = \begin{bmatrix} z_{0\ r} & \cdots & z_{0\ n-1} \\ \vdots & & \vdots \\ z_{r-1\ r} & \cdots & z_{r-1\ n-1} \end{bmatrix}$$

$$U = \begin{bmatrix} u_{0\ 0} & \cdots & u_{0\ r-1} \\ \vdots & & \vdots \\ u_{r-1\ 0} & \cdots & u_{r-1\ r-1} \end{bmatrix} \in GL(r), \quad W = \begin{bmatrix} w_{0\ 0} & \cdots & w_{0\ n-r-1} \\ \vdots & & \vdots \\ w_{r-1\ 0} & \cdots & w_{r-1\ n-r-1} \end{bmatrix}$$

$$A = B_{00}^{-1} \in GL(r), \qquad B = \begin{bmatrix} B_{00} & B_{01} \\ 0 & B_{11} \end{bmatrix} \in \mathrm{PH}_\lambda$$

and $r \times (n-r)$ matrix T includes $N = \{n-(r+1)\}(r-1)$ independent variables $\mathbf{t} = (t_0, \cdots, t_{N-1})$. Then we obtain following five propositions:

Proposition 1. (1) *Natural projection ϕ_2 has following properties:*

- (i) $\phi_2 : Z_\lambda \longrightarrow U_\lambda, Z \longmapsto W(\simeq [I\ W])$ *and ϕ_2 is onto holomorphic.*
- (ii) *For any $W \in U_\lambda$, if we denote $\phi_2^{-1}(W)$ by Σ_W, $\Sigma_W = GL(r)[I\ W]$.*

(2) *Natural projection π has following properties:*

- (i) $\pi : U_\lambda \longrightarrow D_\lambda, W(\simeq [I\ W]) \longmapsto \mathbf{t}(\simeq [I\ T])$ *and π is onto holomorphic.*
- (ii) *For any $\mathbf{t} \in D_\lambda$, if we denote $\pi^{-1}(\mathbf{t})$ by $S_\mathbf{t}$, $S_\mathbf{t} = [I\ T]\mathrm{PH}_\lambda$.*

(3) *Mapping $\pi \circ \phi_2 : Z_\lambda \longrightarrow D_\lambda, Z \longmapsto \mathbf{t}(\simeq [I\ T])$ has following properties:*

- (i) *$\pi \circ \phi_2$ is onto holomorphic.*
- (ii) *For any $\mathbf{t} \in D_\lambda$, $(\pi \circ \phi_2)^{-1}(\mathbf{t}) = \bigcup_{W \in S_\mathbf{t}} \Sigma_W = GL(r)[I\ T]\mathrm{PH}_\lambda$.*

Proposition 2. (1) *There exists biholomorphic mapping*

$$\eta_\lambda : Z_\lambda \xrightarrow{\sim} GL(r) \times U_\lambda, Z = U[I\ W] \longmapsto (U, W)$$

s.t. $\eta_\lambda(\Sigma_W) = GL(r) \times \{W\}$.

(2) *There exists biholomorphic mapping*

$$\zeta_\lambda : Z_\lambda \xrightarrow{\sim} GL(r) \times D_\lambda \times \mathrm{PH}_\lambda, Z = U[I\ W] = UA[I\ T]B \longmapsto (UA, \mathbf{t}, B)$$

s.t. $\zeta_\lambda(\bigcup_{W \in S_\mathbf{t}} \Sigma_W) = GL(r) \times \{\mathbf{t}\} \times \mathrm{PH}_\lambda$.

Proposition 3. *Let $S_A, S_{A,B}$ be sets defined in Definition 7, and let*

$$\mathcal{O}(U_\lambda) = \{\textit{analytic functions defined on } \ U_\lambda\}$$
$$\mathcal{O}(D_\lambda) = \{\textit{analytic functions defined on } \ D_\lambda\}.$$

And suppose that h expresses the function $h(K) = (detK)^{-1}$ $(K \in GL(r))$ and χ_λ expresses the function $\chi_\lambda(L, \alpha)$ $(L \in PH_\lambda)$.

(1) *We can regard $S_A = h{\cdot}\mathcal{O}(U_\lambda)$ by means of $F(Z) = F(U[I\ W]) = h(U)\hat{F}(W)$, where $\hat{F}(W) = F([I\ W])$.*

(2) *We can regard $S_{A,B} = h \cdot \mathcal{O}(D_\lambda) \cdot \chi_\lambda$ by means of $F(Z) = F(UA[I\ T]B) = h(UA)f(\mathbf{t})\chi_\lambda(B, \alpha)$, where $f(\mathbf{t}) = F([I\ T])$.*

Proposition 4. *Suppose $F(Z) \in S_A$ and $F(Z)$ is written as $F(Z) = F(U[I\ W]) = h(U)\hat{F}(W)$, where $\hat{F}(W) \in \mathcal{O}(U_\lambda)$. Then the condition*

$$M_{ij}F = -\delta_{ij}F \quad (0 \le i, j \le r-1)$$

is equivalent to an identity of $h(U)$:

$$h(U)\delta_{i,p}detU = \delta_{i,p}.$$

Therefore, if $F(Z) \in S_A$, $F(Z)$ always satisfies the condition

$$M_{ij}F = -\delta_{ij}F \quad (0 \le i, j \le r-1).$$

Proposition 5. *Suppose $F(Z) \in S_{A,B}$ and $F(Z)$ is written as $F(Z) = F(U[I\ W]) = F(UA[I\ T]B) = h(U)h(A)f(\mathbf{t})\chi_\lambda(B)$. Then the condition*

$$\begin{cases} L_{km}F = \alpha_m^{(k)}F & (0 \le k \le l-1,\ 0 \le m \le \lambda_k - 1) \\ M_{ij}F = -\delta_{ij}F & (0 \le i, j \le r-1) \end{cases}$$

are equivalent to identities of $\chi_\lambda(B)$ and $h(U)$:

$$\begin{cases} \displaystyle\sum_{i=m}^{\lambda_k-1} x_{k,i-m}\frac{\partial\chi_\lambda(B)}{\partial x_{k,i}} = \alpha_m^{(k)}\chi_\lambda(B) \quad (0 \le k \le l-1,\ 0 \le m \le \lambda_k - 1) \\ h(U)\delta_{i,p}detU = \delta_{i,p}. \end{cases}$$

Therefore, if $F(Z) \in S_{A,B}$, $F(Z)$ always satisfies the condition

$$\begin{cases} L_{km}F = \alpha_m^{(k)}F & (0 \le k \le l-1,\ 0 \le m \le \lambda_k - 1) \\ M_{ij}F = -\delta_{ij}F & (0 \le i, j \le r-1). \end{cases}$$

Remark 1. *In the proof of Proposition 5, we will show*

$$\sum_{i=m}^{\lambda_k-1} x_{k,i-m}\frac{\partial\chi_\lambda(B)}{\partial x_{k,i}} = \alpha_m^{(k)}\chi_\lambda(B) \quad (0 \le k \le l-1,\ 0 \le m \le \lambda_k - 1)$$

are identical.

From the above propositions, we can obtain expressions of GCHS on U_λ and on D_λ. Let $\lambda = (\lambda_0, \cdots, \lambda_{l-1})$ be a Young tableau and let $\alpha = (\alpha^{(0)}, \cdots, \alpha^{(l-1)})$ $(\in \mathbf{C}^n)$ be constants which satisfy the condition $\alpha^{(k)} = (\alpha_0^{(k)}, \cdots, \alpha_{\lambda_k - 1}^{(k)}) \in \mathbf{C}^{\lambda_k}$ and $\alpha_0^{(0)} + \cdots + \alpha_0^{(l-1)} = -r$ $(0 < r < n)$. For λ, α and $\tilde{F}(W) (\in \mathcal{O}(U_\lambda))$, let us consider the system defined on U_λ:

$$C_{\lambda,\alpha} \begin{cases} (1) \quad \hat{L}_{km}\hat{F} = \alpha_m^{(k)}\hat{F} & (\text{if } r \leq A_k) \\ (2) \quad I_{km}\hat{F} = \alpha_m^{(k)}\hat{F} & (\text{if } A_k < r \leq A_k + m) \\ (3) \quad J_{km}\hat{F} = \{\delta_{m0}(r - A_k) + \alpha_m^{(k)}\}\hat{F} & (\text{if } A_k + m < r \leq A_{k+1} - 1) \\ (4) \quad K_{km}\hat{F} = -\{\delta_{m0}\lambda_k + \alpha_m^{(k)}\}\hat{F} & (\text{if } A_{k+1} \leq r) \\ (5) \quad \hat{\Box}_{ijpq}\hat{F} = 0 & (0 \leq i, j \leq r - 1,\ 0 \leq p, q \leq n - r - 1) \end{cases}$$

where

$$\hat{L}_{km} = \sum_{q=0}^{r-1} \sum_{p=A_k+m-r}^{A_{k+1}-1-r} w_{q,p-m} \frac{\partial}{\partial w_{q,p}}$$

$$I_{km} = \begin{cases} \displaystyle\sum_{p=A_k+m}^{r+m-1} \frac{\partial}{\partial w_{p-m,p-r}} + \sum_{p=r+m}^{A_{k+1}-1} \sum_{q=0}^{r-1} w_{q,p-m-r} \frac{\partial}{\partial w_{q,p-r}} & (\text{if } r + m < A_{k+1}) \\ \displaystyle\sum_{p=A_k+m}^{A_{k+1}-1} \frac{\partial}{\partial w_{p-m,p-r}} & (\text{if } r + m \geq A_{k+1}) \end{cases}$$

$$J_{km} = \begin{cases} \displaystyle -\sum_{p=A_k+m}^{r-1} \sum_{q=0}^{n-r-1} w_{p,q} \frac{\partial}{\partial w_{p-m,q}} + \sum_{p=r}^{r+m-1} \frac{\partial}{\partial w_{p-m,p-r}} \\ \displaystyle \qquad + \sum_{p=r+m}^{A_{k+1}-1} \sum_{q=0}^{r-1} w_{q,p-m-r} \frac{\partial}{\partial w_{q,p-r}} & (\text{if } r + m < A_{k+1}) \\ \displaystyle -\sum_{p=A_k+m}^{r-1} \sum_{q=0}^{n-r-1} w_{p,q} \frac{\partial}{\partial w_{p-m,q}} + \sum_{p=r}^{A_{k+1}-1} \frac{\partial}{\partial w_{p-m,p-r}} \\ & (\text{if } r + m \geq A_{k+1}) \end{cases}$$

$$K_{km} = \sum_{p=A_k+m}^{A_{k+1}-1} \sum_{q=0}^{n-r-1} w_{p,q} \frac{\partial}{\partial w_{p-m,q}}$$

$$\hat{\Box}_{ijpq} = \frac{\partial^2}{\partial w_{ip} \partial w_{jq}} - \frac{\partial^2}{\partial w_{iq} \partial w_{jp}}$$

δ_{m0} is Kronecker's δ.

Definition 10. *We express the solution space of* $C_{\lambda,\alpha}$ *as*

$$\hat{S} = \{\hat{F}(W) \in \mathcal{O}(U_\lambda) \mid \hat{F}(W) \text{ is a solution of } C_{\lambda,\alpha}\}$$

Theorem 1. *Let* $F(Z) \in S$. *Then* $F(Z)$ *is written as* $F(Z) = F(U[I\ W]) = h(U)\hat{F}(W) \in h \cdot \mathcal{O}(U_\lambda)$ *and* $\hat{F}(W) \in \hat{S}$. *Conversely, let* $\hat{F}(W) \in \hat{S}$. *Then the function* $F(Z)$ $(\in \mathcal{O}(Z_\lambda))$ *defined by* $F(Z) = F(U[I\ W]) = h(U)\hat{F}(W)$ *satisfies* $F(Z) \in S$. *In this sense, the system* $G_{\lambda,\alpha}$ *and the system* $C_{\lambda,\alpha}$ *are equivalent.*

Remark 2. *The corresponding condition to* $M_{ij}F = -\delta_{ij}F$ $(0 \leq i, j \leq r-1)$ *vanishes in* $C_{\lambda,\alpha}$*, and (1),(2),(3) and (4) in* $C_{\lambda,\alpha}$ *are derived from* $L_{km}F = \alpha_m^{(k)}F \quad (0 \leq k \leq l-1,\ 0 \leq m \leq \lambda_k - 1)$*.(5) is derived from* $\Box_{ijpq}F = 0 \quad (0 \leq i, j \leq r-1,\ 0 \leq p, q \leq n-1)$*.*

From Theorem 1, we can say that $C_{\lambda,\alpha}$ is the GCHS on U_λ.

Further we consider a system defined on $GL(r) \times D_\lambda \times \mathrm{PH}_\lambda$. For λ, α and $f(\mathbf{t})(\in \mathcal{O}(D_\lambda))$, let us consider a system

$$\widetilde{H}_{\lambda,\alpha} : (\zeta_\lambda)_*(\Box_{ijpq})h(UA)f(\mathbf{t})\chi_\lambda(B,\alpha) = 0$$
$$(0 \leq i, j \leq r-1,\ 0 \leq p, q \leq n-1).$$

Theorem 2. *Let* $F(Z) \in S$*. Then* $F(Z)$ *is written as* $F(Z) = F(UA[I\ T]B) = h(UA)f(\mathbf{t})\chi_\lambda(B,\alpha) \in h{\cdot}\mathcal{O}(U_\lambda){\cdot}\chi_\lambda$ *and* $h(UA)f(\mathbf{t})\chi_\lambda(B,\alpha)$ *satisfies* $\widetilde{H}_{\lambda,\alpha}$*. Conversely, suppose* $h(UA)f(\mathbf{t})\chi_\lambda(B,\alpha)$ *satisfies* $\widetilde{H}_{\lambda,\alpha}$*. Then the function* $F(Z)$ $(\in \mathcal{O}(Z_\lambda))$ *defined by* $F(Z) = F(UA[I\ T]B) = h(UA)f(\mathbf{t})\chi_\lambda(B,\alpha)$ *satisfies that* $F(Z) \in S$*. In this sense, the system* $G_{\lambda,\alpha}$ *and the system* $\widetilde{H}_{\lambda,\alpha}$ *are equivalent.*

Remark 3. *Corresponding conditions to*

$$\begin{cases} L_{km}F = \alpha_m^{(k)}F & (0 \leq k \leq l-1,\ 0 \leq m \leq \lambda_k - 1) \\ M_{ij}F = -\delta_{ij}F & (0 \leq i, j \leq r-1) \end{cases}$$

vanish in $\widetilde{H}_{\lambda,\alpha}$*.* $\widetilde{H}_{\lambda,\alpha}$ *is derived only from* $\Box_{ijpq}F = 0 \quad (0 \leq i, j \leq r-1,\ 0 \leq p, q \leq n-1)$*.*

Conjecture. $\widetilde{H}_{\lambda,\alpha}$ *is equivalent to a system only for* $f(\mathbf{t})$*. If we denote this system as* $H_{\lambda,\alpha}$*,* $G_{\lambda,\alpha}$ *is equivalent to* $H_{\lambda,\alpha}$*. We may call* $H_{\lambda,\alpha}$ *the GCHS on* D_λ*.*

In the next section we give proofs of the above results.

3 Proofs of results

3.1 Proofs of Lemma 1 and Lemma 2

Proof of Lemma 1. (1) Let $\mathbf{s} = (s_0, \cdots, s_{r-1}) \in \mathbf{P}^{r-1}$, $Z = [I\ W] \in U$, and consider a mapping

$$\tau : \mathbf{P}^{r-1} \times U \longrightarrow F_U, (\mathbf{s}, Z) \longmapsto (<\mathbf{s}Z>, <Z_0, \cdots, Z_{r-1}>)$$

where $Z = [I\ W] = [Z_0 \cdots Z_{r-1}]^t$, $<Z_0, \cdots, Z_{r-1}>$ is a r-dim linear subspace spanned by $Z_0, \cdots, Z_{r-1}(\in \mathbf{C}^n)$ and $<\mathbf{s}Z>$ is a 1-dim linear subspace spanned by $\mathbf{s}Z(\in \mathbf{C}^n)$.

(τ is well defined)

$< Z_0, \cdots, Z_{r-1} >$ is a r-dim linear subspace in $\mathbf{C}^n$. As $Z = [I\ W]$ is rank r, $\mathbf{s}Z \neq 0$. So $< \mathbf{s}Z >$ is a 1-dim subspace in $\mathbf{C}^n$. From $\mathbf{s}Z = (s_0, \cdots, s_{r-1})[Z_0 \cdots Z_{r-1}]^t = s_0 Z_0 + \cdots + s_{r-1} Z_{r-1}$, $< \mathbf{s}Z >$ is included in $< Z_0, \cdots, Z_{r-1} >$. If $\mathbf{u} \sim \mathbf{s}$, then $\mathbf{u} = k\mathbf{s} \quad (k \neq 0)$, and $\mathbf{u}Z = k\mathbf{s}Z$. Then $< \mathbf{u}Z > = < \mathbf{s}Z >$. Therefore $(< \mathbf{s}Z >, < Z_0, \cdots, Z_{r-1} >) \in F_{1,r}$. And $\phi(< \mathbf{s}Z >, < Z_0, \cdots, Z_{r-1} >) = < Z_0 \cdots Z_{r-1} > \in U$. Then $(< \mathbf{s}Z >, < Z_0, \cdots, Z_{r-1} >) \in F_U$.

(τ is bijective)

Suppose $(l, S_r) \in F_U$, $\phi(l, S_r) = S_r \in U$. Then there uniquely exists $[I\ W] = [Z_0 \cdots Z_{r-1}]^t$, s.t. $S_r = < Z_0, \cdots, Z_{r-1} >$, and there uniquely exists $\mathbf{s} \in \mathbf{P}^{r-1}$, s.t. $l = < \mathbf{s}[Z_0 \cdots Z_{r-1}]^t >$. Therefore there uniquely exists $(\mathbf{s}, [I\ W]) \in \mathbf{P}^{r-1} \times U$, s.t. $\tau(\mathbf{s}, [I\ W]) = (l, S_r)$.

(τ is biholomorphic)

By the definition of τ, this property is apparent.

(2) From (1), we have

$$F_U = \{(< \mathbf{s}Z >, < Z_0, \cdots, Z_{r-1} >) | \mathbf{s} \in \mathbf{P}^{r-1}, Z = [Z_0 \cdots Z_{r-1}]^t = [I\ W]\}$$
$$\psi(F_U) = \{< \mathbf{s}Z > | Z = [I\ W]\} \subset \mathbf{P}^{n-1}$$
$$\mathbf{s}Z = \mathbf{s}[I\ W] = (s_0, \cdots, s_{r-1},\ \mathbf{s}W).$$

If $s_k \neq 0$,

$$(s_0, \cdots, s_{r-1},\ \mathbf{s}W) \sim \left(\frac{s_0}{s_k}, \cdots, 1, \cdots, \frac{s_{r-1}}{s_k}, \frac{\mathbf{s}}{s_k} W \right).$$

Then

$$V = \psi(F_U) = \bigcup_{k=0}^{r-1} V_k \subset \mathbf{P}^{n-1},$$

where

$$V_k = \{(*, \cdots, *, \underset{\widehat{k}}{1}, *, \cdots, *)\}.$$

Therefore

$$V = \mathbf{P}^{n-1} - \{(\underbrace{0, \cdots, 0}_{r}, \underbrace{*, \cdots, *}_{n-r})\}.$$

□

Proof of Lemma 2. When $\lambda = (\lambda_0, \lambda_1, \cdots, \lambda_{l-1}), Z \in Z_\lambda$, Z is divided into l blocks as

$$Z = [\overbrace{\cdots}^{\lambda_0} | \overbrace{\cdots}^{\lambda_1} | \cdots | \overbrace{\cdots \| \cdots}^{\lambda_k} | \cdots | \overbrace{\cdots}^{\lambda_{l-1}}],$$

where at λ_k block there are ν columns in the left side of $\|$ line, and $\lambda_0 + \lambda_1 + \cdots + \lambda_{k-1} + \nu = r$.

Let $\mu = (\lambda_0, \lambda_1, \cdots, \lambda_{k-1}, \nu) \quad (|\mu| = r)$, then μ is one of subtableau of λ. For $Z_\mu = [Z^0 \cdots Z^{r-1}]$, we have $det Z_\mu \neq 0$. Therefore $Z \in \tilde{Z}$, then $Z_\lambda \subset \tilde{Z}$. From this we obtain $\mathbf{P}^{r-1} \times \tilde{Z} \supset \mathbf{P}^{r-1} \times Z_\lambda \supset E_\lambda$. □

3.2 Proofs of Proposition 1, 2 and 3

Proof of Proposition 1. (1) (i) For any $Z = [V\ V'] \in Z_\lambda$, there exsits $W(\in GL(r))$ s.t. $U^{-1}[V\ V'] = [I\ W]$. $[I\ W]$ is uniquely determined by Z, and $[I\ W]$ is the representative of the element $\phi_2(Z)$ in $Gr(r,n)$. Then $\phi_2(Z) = W(\simeq [I\ W])$. As $U = V$ and $W = V^{-1}V'$, ϕ_2 is holomorphic.

(ii) If $Z \in GL(r)[I\ W]$, Z is written as $Z = U[I\ W]$ $(U \in GL(r))$. By the property (i) of ϕ_2, $Z \in \phi_2^{-1}(W)$. Then $GL(r)[I\ W] \subset \phi_2^{-1}(W)$. Conversely, if $Z \in \phi_2^{-1}(W)$, there exists $U \in GL(r)$ s.t. $Z = U[I\ W]$. Then $Z \in GL(r)[I\ W]$ and $\phi_2^{-1}(W) \subset GL(r)[I\ W]$. Therefore it holds that $\phi_2^{-1}(W) = GL(r)[I\ W]$.

(2) (i) As Z_λ is open in $M_0(r,n)$ and ϕ_2 is open mapping, U_λ is open in $Gr(r,n)$ and $dim U_\lambda = r(n-r)$. U_λ is PH_λ-invariant and $dim\mathrm{PH}_\lambda = n-1$. So $dim D_\lambda = r(n-r) - (n-1) = \{n-(r+1)\}(r-1)$. Then for any $[I\ W](\in U_\lambda)$, there exist $A \in GL(r), B \in \mathrm{PH}_\lambda$ s.t. $A^{-1}[I\ W]B^{-1} = [I\ T]$, T includes $N = \{n-(r+1)\}(r-1)$ independent variables $\mathbf{t} = (t_0, \cdots, t_N)$. Therefore $\pi(W) = \mathbf{t}(\simeq [I\ T])$. From $T = A^{-1}WB^{-1}$, π is holomorphic.

(ii) If $[I\ W] \in [I\ T]\mathrm{PH}_\lambda$, $[I\ W]$ is written as $[I\ W] = A[I\ T]B$

$$\left(A \in GL(r), \quad B = \begin{bmatrix} B_{00} & B_{01} \\ 0 & B_{11} \end{bmatrix} \in \mathrm{PH}_\lambda\right)$$

where $B_{00} \in GL(r)$ and A is needed for the adjustment of $AB_{00} = I$. From this, $[I\ T]\mathrm{PH}_\lambda \subset \pi^{-1}(\mathbf{t})$. Conversely, if $[I\ W] \in \pi^{-1}(\mathbf{t})$, there exist $A \in GL(r)$, $B \in \mathrm{PH}_\lambda$ s.t. $[I\ W] = A[I\ T]B$. As $A[I\ T]B \in [I\ T]\mathrm{PH}_\lambda$, $\pi^{-1}(\mathbf{t}) \subset [I\ T]\mathrm{PH}_\lambda$. Therefore, we have $\pi^{-1}(\mathbf{t}) = [I\ T]\mathrm{PH}_\lambda$.

(3) (i) From (1)(i) and (2)(i), it is apparent.

(ii) $(\pi \circ \phi_2)^{-1}(\mathbf{t}) = \phi_2^{-1}(\pi^{-1}(\mathbf{t})) = \phi_2^{-1}(S_{\mathbf{t}}) = \bigcup_{W \in S_{\mathbf{t}}} \phi_2^{-1}(W) = \bigcup_{W \in S_{\mathbf{t}}} \Sigma_W$. On the other hand, $\phi_2^{-1}(\pi^{-1}(\mathbf{t})) = \phi_2^{-1}([I\ T]\mathrm{PH}_\lambda) = GL(r)[I\ T]\mathrm{PH}_\lambda$.

□

Proof of Proposition 2. (1) Let define mappings:

$$\begin{aligned} &\eta_\lambda : Z_\lambda \longrightarrow GL(r) \times U_\lambda, Z = U[I\ W] \longmapsto (U, W) \\ &\nu_\lambda : GL(r) \times U_\lambda \longrightarrow Z_\lambda, (U, W) \longmapsto Z = U[I\ W] \end{aligned}$$

η_λ and ν_λ are well defined and holomorphic. Apparently ν_λ is inverse of η_λ, so η_λ is biholomorphic mapping.

From Proposition 1 (1)(ii), $\eta_\lambda(\Sigma_\lambda) = \eta_\lambda(GL(r)[I\ W]) = GL(r) \times W$.

(2) Let define mappings:

$$\begin{aligned} &\zeta_\lambda : Z_\lambda \longrightarrow GL(r) \times D_\lambda \times \mathrm{PH}_\lambda, Z = U[I\ W] = UA[I\ T]B \longmapsto (UA, \mathbf{t}, B) \\ &\xi_\lambda : GL(r) \times D_\lambda \times \mathrm{PH}_\lambda \longrightarrow Z_\lambda, (C, \mathbf{t}, B) \longmapsto Z = C[I\ T]B \end{aligned}$$

ζ_λ and ξ_λ are well defined and holomorphic. Apparently ξ_λ is inverse of ζ_λ, so ζ_λ is biholomorphic mapping. From Proposition 1(3)(ii), $\zeta_\lambda(\bigcup_{W \in S_{\mathbf{t}}} \Sigma_W) = \zeta_\lambda(GL(r)[I\ T]\mathrm{PH}_\lambda) = GL(r) \times \{\mathbf{t}\} \times \mathrm{PH}_\lambda$.

□

Proof of Proposition 3. (1) If $F(Z) \in S_A$,

$$F(Z) = F(U[I\ W]) = h(U)F([I\ W]) = h(U)\hat{F}(W) \in h \cdot \mathcal{O}(U_\lambda).$$

Then $S_A \subset h \cdot \mathcal{O}(U_\lambda)$. Conversely, if $h(U)\hat{F}(W) \in h \cdot \mathcal{O}(U_\lambda)$, we can define an analytic function $\tilde{F}(Z)$ as

$$\tilde{F}(Z) = \tilde{F}(U[I\ W]) = h(U)\hat{F}(W).$$

Since any $Z (\in Z_\lambda)$ can be written as $Z = U[I\ W]$, $\tilde{F}(Z)$ is well defined. For any $K \in GL(r)$,

$$\tilde{F}(KZ) = \tilde{F}(KU[I\ W]) = h(KU)\hat{F}(W) = h(K)h(U)\hat{F}(W) = h(K)\tilde{F}(Z).$$

Then $\tilde{F}(Z) \in S_A$, and $h \cdot \mathcal{O}(U_\lambda) \subset S_A$. From these, we obtain $S_A = h \cdot \mathcal{O}(U_\lambda)$.

(2) If $F(Z) \in S_{A,B}$,

$$\begin{aligned} F(Z) &= F(U[I\ W]) = F(UA[I\ T]B) = h(UA)F([I\ T])\chi_\lambda(B, \alpha) \\ &= h(UA)f(\mathbf{t})\chi_\lambda(B, \alpha) \in h \cdot \mathcal{O}(D_\lambda) \cdot \chi_\lambda. \end{aligned}$$

Then $S_{A,B} \subset h \cdot \mathcal{O}(D_\lambda) \cdot \chi_\lambda$. Conversely, if $h(V)f(\mathbf{t})\chi_\lambda(B, \alpha) \in h \cdot \mathcal{O}(D_\lambda) \cdot \chi_\lambda$, we can define an analytic function $\tilde{F}(Z)$ on Z_λ as

$$\tilde{F}(Z) = h(V)f(\mathbf{t})\chi_\lambda(B, \alpha),$$

because that $\zeta_\lambda : Z_\lambda \longrightarrow GL(r) \times D_\lambda \times \mathrm{PH}_\lambda$ is biholomorphic. By similar calculations in (1), we can check that $\tilde{F}(Z) \in S_{A,B}$. So $h \cdot \mathcal{O}(D_\lambda) \cdot \chi_\lambda \subset S_{A,B}$. Therefore we have proved that $S_{A,B} = h \cdot \mathcal{O}(D_\lambda) \cdot \chi_\lambda$. □

3.3 Proofs of Proposition 4 and 5

To give a proof of Proposition 4, we first prepare a Lemma.

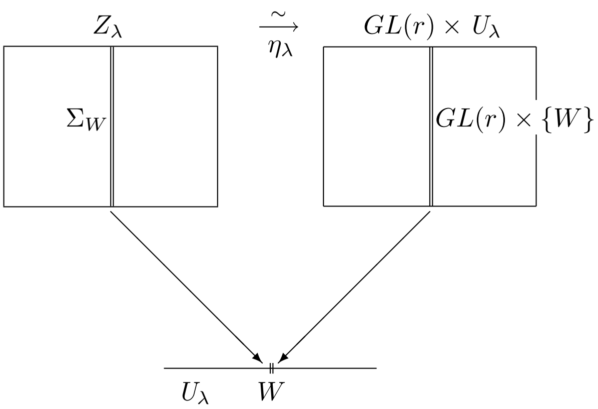
Lemma 3. *Suppose* $U = [u_{ij}] \in GL(r)$, $h(U) = (detU)^{-1}$ *and* Δ_{ij} *be the cofactor of* u_{ij} *in* U. *Then we get*

(1) $\dfrac{\partial^2 h}{\partial u_{ij}\partial u_{\bar{i}\bar{j}}} = \dfrac{\partial^2 h}{\partial u_{i\bar{j}}\partial u_{\bar{i}j}}$

(2) $\dfrac{\partial h}{\partial u_{ij}} = -h^2\Delta_{ij}$

This is a Lemma given in Horikawa [6]. So we omit the proof.

Proof of Proposition 4. By Proposition 2(1), we have the following fibrations:



Suppose $Z \in Z_\lambda$ and that s belongs to an open neighborhood of $0 \in \mathbf{C}$. For any $A \in M(r,r)$, $exp(sA)$ is an element in $GL(r)$. So we obtain $F(exp(sA)Z) = exp(-sTrA)F(Z)$. If $Z \in \Sigma_W$, then $exp(sA)Z$ is a curve in Σ_W. Let us consider the differential $\frac{d}{ds}F(exp(sA)Z)\mid_{s=0}$. Here we obtain

$$\frac{d}{ds}F(exp(sA)Z)\Big|_{s=0} = \frac{d}{ds}exp(-sTrA)F(Z)\Big|_{s=0}. \tag{1}$$

Then, if $A = [a_{ij}]$ and $Z = [z_{ij}]$, (1) means

$$\begin{aligned} \sum_{i,j} \frac{\partial F}{\partial z_{ij}}(Z) \sum_p a_{ip} z_{pj} &= (-\sum_{i=0}^{r-1} a_{ii})F(Z) \\ \sum_{i,p} a_{ip} \sum_j z_{pj} \frac{\partial F}{\partial z_{ij}} &= \sum_{i,p} a_{ip}(-\delta_{ip})F. \end{aligned}$$

Since A is any matrix, this relation is equivalent to

$$\sum_{j=0}^{n-1} z_{pj} \frac{\partial F}{\partial z_{ij}} = -\delta_{ip}F \qquad (0 \le i, p \le r-1).$$

Changing indices,

$$\sum_{p=0}^{n-1} z_{ip} \frac{\partial F}{\partial z_{jp}} = -\delta_{ji}F \qquad (0 \le i, j \le r-1)$$

$$M_{ij}F = -\delta_{ij}F \qquad (0 \le i, j \le r-1). \tag{2}$$

Therefore, the condition (2) is equivalent to (1).

Here we set as

$$F(Z) = F(U[I\ W]) = h(U)\hat{F}(W) = \bar{F}(U,W).$$

Then we have

$$F(exp(sA)Z) = F(exp(sA)U[I\ W]) = \bar{F}(exp(sA)U, W).$$

We note that $(exp(sA)U, W)$ is a curve in the fiber $GL(r) \times \{W\}$ in $GL(r) \times U_\lambda$. If we use $\bar{F}$, (1) is rewritten as

$$\frac{d}{ds}\bar{F}(exp(sA)U, W)\Big|_{s=0} = \frac{d}{ds}exp(-sTrA)\bar{F}(U,W)\Big|_{s=0}. \tag{3}$$

Then

$$\begin{aligned} \sum_{i,j} \frac{\partial \bar{F}}{\partial u_{ij}}(U,W) \sum_{p=0}^{r-1} a_{ip} u_{pj} &= (-\sum_{i=0}^{r-1} a_{ii})\bar{F}(U,W) \\ \sum_{i,p} a_{ip} \sum_j u_{pj} \frac{\partial \bar{F}}{\partial u_{ij}} &= \sum_{i,p} a_{ip}(-\delta_{ip})\bar{F}. \end{aligned}$$

Since A is any matrix, this relation is equivalent to

$$\sum_{j=0}^{r-1} u_{pj} \frac{\partial \bar{F}}{\partial u_{ij}} = -\delta_{ip}\bar{F} \qquad (0 \le i, p \le r-1). \tag{4}$$

This is a equation for $\bar{F}$. As $\bar{F}(U,W) = h(U)\hat{F}(W)$, (4) is

$$\hat{F}(W)\sum_{j=0}^{r-1} u_{pj}\frac{\partial h(U)}{\partial u_{ij}} = \hat{F}(W)(-\delta_{ip}h(U)). \tag{5}$$

If $\hat{F}(W) = 0$, then this equation is an identity. So it is equivalent to any identity of $h(U)$. When $\hat{F}(W) \neq 0$, (5) is equivalent to

$$\sum_{j=0}^{r-1} u_{pj}\frac{\partial h(U)}{\partial u_{ij}} = -\delta_{ip}h(U) \qquad (0 \leq i, p \leq r-1). \tag{6}$$

By Lemma 3 (2), (6) is equivalent to

$$\sum_{j=0}^{r-1} u_{pj}(-h^2\Delta_{ij}) = -\delta_{ip}h$$

$$h\sum_{j=0}^{r-1} u_{pj}\Delta_{ij} = h\,\delta_{ip}\,detU = \delta_{ip}. \tag{7}$$

The last equation is an identity. So the condition (5) is equivalent to the identity (7). From these, the condition (2) is equivalent to the identity (7). ☐

Remark 4. *Horikawa [6] proved Proposition 4 for the case $F(Z) \in S$ with $\lambda = (1, 1, \cdots, 1)$ by direct calculation. We can also prove Proposition 4 of general case by direct calculation. Actually, equation $M_{ij}F = -\delta_{ij}F$ is rewritten into (6) by the change of variables $Z = U[I\ W]$.*

To prove Proposition 5, next we prepare a Lemma on functions θ_j.

Lemma 4. *Suppose λ_k is an component of $\lambda = (\lambda_0, \cdots, \lambda_{l-1})$, and $\mathbf{x} = (x_0, x_1, \cdots, x_{\lambda_k - 1})$ $(x_0 \neq 0)$ are variables. For $\theta_j(x_0, x_1, \cdots, x_j)$ and any integer m $(0 \leq m \leq \lambda_k - 1)$, it holds that*

$$\begin{cases} x_0\dfrac{\partial}{\partial x_m}\theta_m = 1 \\ \\ (x_0\dfrac{\partial}{\partial x_m} + x_1\dfrac{\partial}{\partial x_{m+1}} + \cdots + x_i\dfrac{\partial}{\partial x_{m+i}})\theta_{m+i} = 0 \quad (1 \leq i \leq \lambda_k - m - 1). \end{cases}$$

Proof. We define a function $\Phi(s)$ as

$$\begin{aligned} \Phi(s) &= log[x_0 + x_1T + \cdots + x_{m-1}T^{m-1} + (x_m + sx_0)T^m \\ &\qquad +(x_{m+1} + sx_1)T^{m+1} + \cdots + (x_{\lambda_k-1} + sx_{\lambda_k-1-m})T^{\lambda_k-1}] \\ &= log[x_0 + x_1T + \cdots + x_{\lambda_k-1}T^{\lambda_k-1} \\ &\qquad +s(x_0T^m + x_1T^{m+1} + \cdots + x_{\lambda_k-1-m}T^{\lambda_k-1})] \\ &= \theta_0(x_0) + \theta_1(x_0, x_1)T + \cdots + \theta_m(x_0, \cdots, x_{m-1}, x_m + sx_0)T^m \\ &\qquad +\theta_{m+1}(x_0, \cdots, x_m + sx_0, x_{m+1} + sx_1)T^{m+1} \\ &\qquad \cdots \\ &\qquad +\theta_{m+i}(x_0, \cdots, x_m + sx_0, \cdots, x_{m+i} + sx_i)T^{m+i} \\ &\qquad \cdots \\ &\qquad +\theta_{\lambda_k-1}(x_0, \cdots, x_m + sx_0, \cdots, x_{\lambda_k-1} + sx_{\lambda_k-1-m})T^{\lambda_k-1} \\ &\qquad \cdots, \end{aligned}$$

where s is a complex parameter defined in an open neighborhood of $0 \in \mathbf{C}$. Then we have

$$\begin{aligned}
\frac{d\Phi}{ds}(s)\Big|_{s=0} &= \left(\frac{\partial\theta_m}{\partial x_m}x_0\right)T^m \\
&\quad + \left(x_0\frac{\partial}{\partial x_m} + x_1\frac{\partial}{\partial x_{m+1}}\right)\theta_{m+1}T^{m+1} \\
&\quad \cdots \\
&\quad + \left(x_0\frac{\partial}{\partial x_m} + x_1\frac{\partial}{\partial x_{m+1}} + \cdots + x_i\frac{\partial}{\partial x_{m+i}}\right)\theta_{m+i}T^{m+i} \\
&\quad \cdots \\
&\quad + \left(x_0\frac{\partial}{\partial x_m} + x_1\frac{\partial}{\partial x_{m+1}} + \cdots + x_{\lambda_k-1-m}\frac{\partial}{\partial x_{\lambda_k-1}}\right)\theta_{\lambda_k-1}T^{\lambda_k-1} \\
&\quad \cdots \\
&= \frac{T^m[1+\frac{x_1}{x_0}T+\cdots+\frac{x_{\lambda_k-1-m}}{x_0}T^{\lambda_k-1-m}]}{1+\frac{x_1}{x_0}T+\cdots\frac{x_{\lambda_k-1}}{x_0}T^{\lambda_k-1}}\cdots(*).
\end{aligned}$$

Here we set $S = \frac{x_1}{x_0}T + \cdots \frac{x_{\lambda_k-1}}{x_0}T^{\lambda_k-1}$. Then

$$\begin{aligned}
(*) \quad &= T^m\left[1 - S + S^2 - S^3 + \cdots\right]\left[1 + S - \left(\frac{x_{\lambda_k-m}}{x_0}T^{\lambda_k-m} + \cdots + \frac{x_{\lambda_k-1}}{x_0}T^{\lambda_k-1}\right)\right] \\
&= T^m - \left(\frac{x_{\lambda_k-m}}{x_0}T^{\lambda_k} + \cdots + \frac{x_{\lambda_k-1}}{x_0}T^{\lambda_k+m-1}\right)(1+O(T)) \\
&= T^m - O(T^{\lambda_k}).
\end{aligned}$$

Therefore, we obtain

$$\begin{cases} x_0\dfrac{\partial}{\partial x_m}\theta_m = 1 \\ \left(x_0\dfrac{\partial}{\partial x_m} + x_1\dfrac{\partial}{\partial x_{m+1}} + \cdots + x_i\dfrac{\partial}{\partial x_{m+i}}\right)\theta_{m+i} = 0 \quad (1 \le i \le \lambda_k - m - 1). \end{cases}$$

□

Proof of Proposition 5. As in the proof of Proposition 4,

$$M_{ij}F = -\delta_{ij}F \quad (0 \le i, j \le r-1)$$

is equivalent to the condition

$$\frac{d}{ds}F(exp(sC)Z)\Big|_{s=0} = \frac{d}{ds}exp(-sTrC)F(Z)\Big|_{s=0},$$

where $C \in M(r,r)$ and s belongs to an open neighborhood of $0 \in \mathbf{C}$. Since $F(Z) = F(UA[I\ T]B) = h(U)h(A)f(\mathbf{t})\chi_\lambda(B,\alpha)$ and A depends on B, we can set $F(Z) = \bar{F}(U,\mathbf{t},B)$. Then the above equation is written as

$$\frac{d}{ds}\bar{F}(exp(sC)U,\mathbf{t},B)\Big|_{s=0} = \frac{d}{ds}exp(-sTrC)\bar{F}(U,\mathbf{t},B)\Big|_{s=0}.$$

This is equivalent to

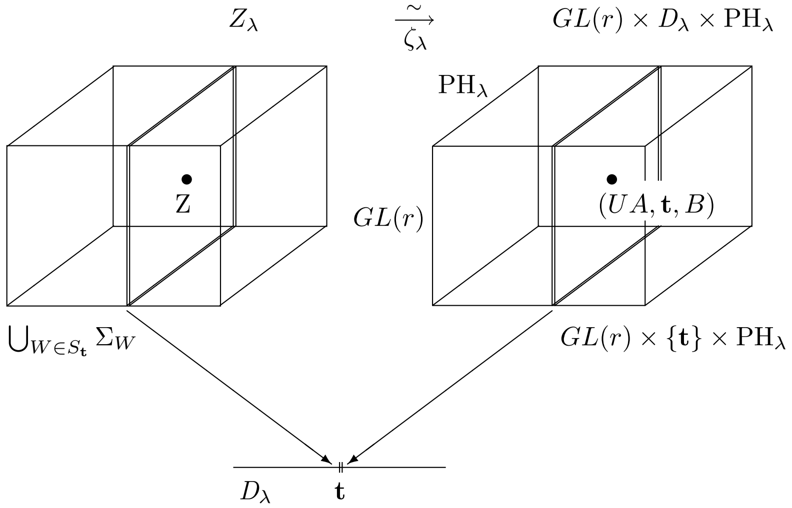
$$\sum_{j=0}^{r-1}u_{pj}\frac{\partial\bar{F}}{\partial u_{ij}} = -\delta_{ip}\bar{F} \quad (0 \le i, p \le r-1)$$

and we can obtain the equivalent condition

$$h(U)\delta_{ip}detU = \delta_{ip}$$

by similar calculations in the proof of Proposition 4.

Next, by Proposition 2 (2), we have the following fibration:



Suppose Z belongs to $\bigcup_{w\in S_\mathbf{t}} \Sigma_W = GL(r)[I\ T]\mathrm{PH}_\lambda$. We note that Z is written as $Z = U[I\ W] = UA[I\ T]B$, where $UA \in GL(r), B = B_0 \bigoplus B_1 \bigoplus \cdots \bigoplus B_k \bigoplus \cdots \bigoplus B_{l-1} \in \mathrm{PH}_\lambda$. Here we set

$$\begin{array}{l} h_{k,m} = I_0 \bigoplus I_1 \bigoplus \cdots \bigoplus H_{k,m} \bigoplus \cdots \bigoplus I_{l-1} \in \mathrm{PH}_\lambda \\ H_{k,m} = I_k + s\Lambda^m \\ (0 \le k \le l-1,\ 0 \le m \le \lambda_k - 1) \end{array}$$

where I_j $(0 \le j \le l-1)$ is a unit matrix of size λ_j, Λ is the shift matrix of size λ_k and s is a complex parameter defined in an open neighborhood of $0 \in \mathbf{C}$. Then $Zh_{k,m}$ parametrized by s is a curve in $\bigcup_{W\in S_\mathbf{t}} \Sigma_W$. Let us consider the differential $\frac{d}{ds}F(Zh_{k,m}) \mid_{s=0}$. From the property of F, we obtain

$$\frac{d}{ds}F(Zh_{k,m}) \Big|_{s=0} = \frac{d}{ds}F(Z)\chi_\lambda(h_{k,m}, \alpha) \Big|_{s=0} . \tag{8}$$

Since

$$\begin{array}{rl} Zh_{k,m} & = [Z^0 \cdots Z^k \cdots Z^{l-1}](I_0 \bigoplus I_1 \bigoplus \cdots \bigoplus H_{k,m} \bigoplus \cdots \bigoplus I_{l-1}) \\ & = [Z^0 \cdots Z^{k-1}\ \ \tilde{Z}^k(s)\ \ Z^{k+1} \cdots Z^{l-1}], \end{array}$$

where

$$Z^k = \begin{bmatrix} z_{0,A_k} & \cdots & z_{0,A_k+m-1} & z_{0,A_k+m} & \cdots & z_{0,A_k+\lambda_k-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ z_{r-1,A_k} & \cdots & z_{r-1,A_k+m-1} & z_{r-1,A_k+m} & \cdots & z_{r-1,A_k+\lambda_k-1} \end{bmatrix},$$

$$\tilde{Z}^k(s) = \left[\begin{array}{lllll} z_{0,A_k} & \cdots & z_{0,A_k+m-1} & sz_{0,A_k} + z_{0,A_k+m} & \cdots \\ \vdots & & \vdots & \vdots & \\ z_{r-1,A_k} & \cdots & z_{r-1,A_k+m-1} & sz_{r-1,A_k} + z_{r-1,A_k+m} & \cdots \end{array} \right.$$

$$\left.\begin{matrix} sz_{0,A_k+\lambda_{k-1}-1-m}+z_{0,A_k+\lambda_k-1} \\ \vdots \\ sz_{r-1,A_k+\lambda_{k-1}-1-m}+z_{r-1,A_k+\lambda_k-1} \end{matrix}\right] \qquad (A_k=\lambda_0+\lambda_1+\cdots\lambda_{k-1}),$$

then we have

$$\frac{d}{ds}F(Zh_{k,m})\Big|_{s=0} = \sum_{q=0}^{r-1}\sum_{p=A_k+m}^{A+\lambda_k-1} z_{q,p-m}\frac{\partial F}{\partial z_{q,p}}(Z) = L_{k,m}F.$$

On the other hand, if $0<m\leq\lambda_k-1$,

$$\begin{aligned}\chi_\lambda(h_{k,m},\alpha) &= \chi_{\lambda_0}(I_{\lambda_0},\alpha^{(0)})\cdots\chi_{\lambda_{k-1}}(I_{\lambda_{k-1}},\alpha^{(k-1)}) \\ &\qquad \chi_{\lambda_k}(H_{k,m},\alpha^{(k)})\chi_{\lambda_{k+1}}(I_{\lambda_{k+1}},\alpha^{(k+1)})\cdots \\ &= \chi_{\lambda_k}(H_{k,m},\alpha^{(k)}) \\ &= exp\left(\sum_{0\leq i<\lambda_k}\alpha_i^{(k)}\theta_i(1,0,\cdots,0,s,0,\cdots)\right) \\ &= exp\left(\alpha_m^{(k)}s+\alpha_{2m}^{(k)}(-\frac{s^2}{2})+\cdots\right),\end{aligned}$$

if $m=0$,

$$\begin{aligned}\chi_\lambda(h_{k,m},\alpha) &= exp\left(\sum_{0\leq i<\lambda_k}\alpha_i^{(k)}\theta_i(1+s,0,\cdots)\right) \\ &= (1+s)^{\alpha_0^{(k)}}.\end{aligned}$$

So we obtain

$$\frac{d}{ds}F(Z)\chi_\lambda(h_{k,m},\alpha)\Big|_{s=0} = \alpha_m^{(k)}F(Z).$$

Therefore (8) is equivalent to the equation

$$L_{k,m}F=\alpha_m^{(k)}F. \tag{9}$$

Here from $F(Z)$ we define a function $\tilde{F}$ defined on $GL(r)\times D_\lambda\times \mathrm{PH}_\lambda$ by $F(Z)=F(UA[I\ T]B)=h(UA)f(\mathbf{t})\chi_\lambda(B,\alpha)=\tilde{F}(U,\mathbf{t},B)$. We note that A depends on B. And we rewrite the relation (8) into the relation for $\tilde{F}(U,\mathbf{t},B)$. From

$$\begin{aligned}\frac{d}{ds}F(Zh_{k,m})\Big|_{s=0} &= \frac{d}{ds}\tilde{F}(U,\mathbf{t},Bh_{k,m})\Big|_{s=0} \\ \frac{d}{ds}F(Z)\chi_\lambda(h_{k,m},\alpha)\Big|_{s=0} &= \alpha_m^{(k)}F(Z)=\alpha_m^{(k)}\tilde{F}(U,\mathbf{t},B),\end{aligned}$$

we have

$$\frac{d}{ds}\tilde{F}(U,\mathbf{t},Bh_{k,m})\Big|_{s=0} = \alpha_m^{(k)}\tilde{F}(U,\mathbf{t},B). \tag{10}$$

Since

$$\begin{aligned}Bh_{k,m} &= B_0\oplus B_1\oplus\cdots\oplus B_kH_{k,m}\oplus\cdots\oplus B_{l-1} \\ B_kH_{k,m} &= (x_{k,0}\Lambda^0+\cdots+x_{k,\lambda_k-1}\Lambda^{\lambda_k-1})(\Lambda^0+s\Lambda^m) \\ &= x_{k,0}\Lambda^0+\cdots+x_{k,m-1}\Lambda^{m-1} \\ &\quad +(x_{k,m}+sx_{k,0})\Lambda^m+\cdots+(x_{k,\lambda_k-1}+sx_{k,\lambda_k-1-m})\Lambda^{\lambda_k-1},\end{aligned}$$

it holds that

$$\frac{d}{ds}\tilde{F}(U, \mathbf{t}, Bh_{k,m})\Big|_{s=0} = \sum_{i=m}^{\lambda_k-1} x_{k,i-m}\frac{\partial \tilde{F}}{\partial x_{k,i}}.$$

Then (10) is written as

$$\sum_{i=m}^{\lambda_k-1} x_{k,i-m}\frac{\partial \tilde{F}}{\partial x_{k,i}} = \alpha_m^{(k)}\tilde{F} \qquad (0 \le k \le l-1,\ 0 \le m \le \lambda_k - 1) \tag{11}$$

This is an equation for $\tilde{F}$ on $GL(r) \times D_\lambda \times \mathrm{PH}_\lambda$. Since $\tilde{F}(U, \mathbf{t}, B) = h(UA)f(\mathbf{t})$ $\chi_\lambda(B, \alpha)$, $h(UA) = det(UA)^{-1} \neq 0$, if $f(\mathbf{t}) \neq 0$, then (11) is equivalent to

$$\sum_{i=m}^{\lambda_k-1} x_{k,i-m}\frac{\partial \chi_\lambda(B,\alpha)}{\partial x_{k,i}} = \alpha_m^{(k)}\chi_\lambda(B,\alpha) \qquad (0 \le k \le l-1,\ 0 \le m \le \lambda_k - 1) \tag{12}$$

This is an equation on $GL(r) \times D_\lambda \times \mathrm{PH}_\lambda$ for the function $\chi_\lambda(B)$. We note that if $f(\mathbf{t}) = 0$ then (11) is equivalent to any identity.

From now, we will show that (12) is the identical equation.

We can assume there exists a number j $(j \neq k)$ such that diagonal elements of B_j are equal to one in the expression $B = B_0 \oplus \cdots \oplus B_k \oplus \cdots \oplus B_{l-1}$. Since $\chi_\lambda(B,\alpha) = \chi_{\lambda_0}(B_0, \alpha^{(0)}) \cdots \chi_{\lambda_k}(B_k, \alpha^{(k)}) \cdots \chi_{\lambda_{l-1}}(B_{l-1}, \alpha^{(l-1)})$, (12) is equivalent to

$$\sum_{i=m}^{\lambda_k-1} x_{k,i-m}\frac{\partial \chi_{\lambda_k}(B_k,\alpha^{(k)})}{\partial x_{k,i}} = \alpha_m^{(k)}\chi_{\lambda_k}(B_k,\alpha^{(k)}) \tag{13}$$

$$(0 \le k \le l-1,\ 0 \le m \le \lambda_k - 1)$$

As $B_k = [x_{k,0}, \cdots, x_{k,\lambda_k-1}]$,

$$\chi_{\lambda_k}(B_k, \alpha^{(k)}) = exp\left(\alpha_0^{(k)}\theta_0(x_{k,0}) + \cdots + \alpha_{\lambda_k-1}^{(k)}\theta_{\lambda_k-1}(x_{k,0}, \cdots, x_{k,\lambda_k-1})\right)$$

For the simplicity, we omit k in α, x and $\lambda_k, \alpha^{(k)}$ in χ. Then we have

$$\chi(B_k) = exp\left(\alpha_0\theta_0(x_0) + \cdots + \alpha_{\lambda_k-1}\theta_{\lambda_k-1}(x_0, \cdots, x_{\lambda_k-1})\right).$$

Using this notation, (13) is equivalent to the following equation:

$$\begin{aligned}
&\sum_{i=m}^{\lambda_k-1} x_{k,i-m}\frac{\partial \chi(B_k)}{\partial x_{k,i}} \\
&= \left(x_0\frac{\partial}{\partial x_m} + x_1\frac{\partial}{\partial x_{m+1}} + \cdots + x_{\lambda_k-1-m}\frac{\partial}{\partial x_{\lambda_k-1}}\right)\chi(B_k) \\
&= \chi(B_k)\Bigg[x_0\left\{\alpha_m\frac{\partial}{\partial x_m}\theta_m + \alpha_{m+1}\frac{\partial}{\partial x_m}\theta_{m+1} + \cdots + \alpha_{\lambda_k-1}\frac{\partial}{\partial x_m}\theta_{\lambda_k-1}\right\} \\
&\qquad + x_1\left\{\alpha_{m+1}\frac{\partial}{\partial x_{m+1}}\theta_{m+1} + \cdots + \alpha_{\lambda_k-1}\frac{\partial}{\partial x_{m+1}}\theta_{\lambda_k-1}\right\} \\
&\qquad \cdots \\
&\qquad + x_i\left\{\alpha_{m+i}\frac{\partial}{\partial x_{m+i}}\theta_{m+i} + \cdots + \alpha_{\lambda_k-1}\frac{\partial}{\partial x_{m+i}}\theta_{\lambda_k-1}\right\} \\
&\qquad \cdots \\
&\qquad + x_{\lambda_k-1-m}\left\{\alpha_{\lambda_k-1}\frac{\partial}{\partial x_{\lambda_k-1}}\theta_{\lambda_k-1}\right\}\Bigg] \\
&= \alpha_m\chi(B_k)
\end{aligned}$$

And this equation is equivalent to the next equation:

$$\begin{aligned}
&\alpha_m \left(x_0 \frac{\partial}{\partial x_m} \theta_m \right) \\
&+\alpha_{m+1} \left[x_0 \frac{\partial}{\partial x_m} + x_1 \frac{\partial}{\partial x_{m+1}} \right] \theta_{m+1} \\
&\qquad \cdots \\
&+\alpha_{m+i} \left[x_0 \frac{\partial}{\partial x_m} + x_1 \frac{\partial}{\partial x_{m+1}} + \cdots + x_i \frac{\partial}{\partial x_{m+i}} \right] \theta_{m+i} \\
&\qquad \cdots \\
&+\alpha_{\lambda_k - 1} \left[x_0 \frac{\partial}{\partial x_m} + x_1 \frac{\partial}{\partial x_{m+1}} + \cdots + x_{\lambda_k - 1 - m} \frac{\partial}{\partial x_{\lambda_k - 1}} \right] \theta_{\lambda_k - 1} = \alpha_m
\end{aligned} \tag{14}$$

From Lemma 4, this equation (14) is an identity. Therefore, (9) is equivalent to the identity (12). So, we have completed the proof of Proposition 5. ☐

3.4 Preliminaries for the proof of Theorem 1

In this section, we give two lemmas for the proof of Theorem 1. We use the same symbols $h, \Delta_{i,j}$ as in Lemma 3.

Lemma 5. *Let* $\Delta_{i,j}$ *be expanded as*

$$\begin{aligned}
\Delta_{i,j} &= (-1)^{i+j} \begin{array}{c} \qquad j \\ \qquad \vee \\ \left| \begin{array}{cc|cc} u_{0,0} & \cdots & \cdots & u_{0,r-1} \\ \vdots & & & \vdots \\ \hline \vdots & & & \vdots \\ u_{r-1,0} & \cdots & \cdots & u_{r-1,r-1} \end{array} \right| \end{array} < i \\
&= (-1)^{i+j} \left[u_{\bar{i},0} \bar{\Delta}_{\bar{i},0} + u_{\bar{i},1} \bar{\Delta}_{\bar{i},1} + \cdots + u_{\bar{i},r-1} \bar{\Delta}_{\bar{i},r-1} \right].
\end{aligned}$$

We define the symbol $\Delta_{i,j,\bar{i},\bar{j}}$ *as*

$$\Delta_{i,j,\bar{i},\bar{j}} = \begin{cases} 0 & (i = \bar{i} \text{ or } j = \bar{j}) \\ (-1)^{i+j} \bar{\Delta}_{\bar{i},\bar{j}} & (i \neq \bar{i} \text{ and } j \neq \bar{j}). \end{cases}$$

Then we have

(1) $\dfrac{\partial \Delta_{i,j}}{\partial u_{\bar{i},\bar{j}}} = \Delta_{i,j,\bar{i},\bar{j}}$

(2) $(det U) \Delta_{i,j,\bar{i},\bar{j}} = \Delta_{i,j} \Delta_{\bar{i},\bar{j}} - \Delta_{i,\bar{j}} \Delta_{\bar{i},j}$ *(Jacobi's formula)*

(3) $\Delta_{i,\bar{j},\bar{i},j} = -\Delta_{i,j,\bar{i},\bar{j}}$

(4) $\Delta_{i,j,\bar{i},\bar{j}} = \Delta_{\bar{i},\bar{j},i,j}$.

(1),(2) and (3) were given in the proof of Lemma 1 in Horikawa [6]. So we omit the proofs. Here we will give the proof of (4).

Proof of (4). It is sufficient to prove the case $i \neq \bar{i}$ and $j \neq \bar{j}$.

Suppose that $i < \bar{i}$ and $j < \bar{j}$. Let $U_{i,\bar{i},j,\bar{j}}$ be a matrix made from a matrix U by deleting $i, \bar{i}$ rows and $j, \bar{j}$ columns. Then

$$\begin{array}{l} \Delta_{i,j,\bar{i},\bar{j}} = (-1)^{i+j}\bar{\Delta}_{\bar{i},\bar{j}} = (-1)^{i+j+\bar{i}+\bar{j}} detU_{i,\bar{i},j,\bar{j}} \\ \Delta_{\bar{i},\bar{j},i,j} = (-1)^{\bar{i}+\bar{j}}\bar{\Delta}_{i,j} = (-1)^{\bar{i}+\bar{j}+i+j} detU_{i,\bar{i},j,\bar{j}} \end{array}$$

So $\Delta_{i,j,\bar{i},\bar{j}} = \Delta_{\bar{i},\bar{j},i,j}$.

By similar consideration, we get the same result for other cases. □

From the relation $Z = [V\ V'] = U[I\ W]$, we obtain

$$\begin{cases} V = U \\ V' = UW \end{cases} \qquad \begin{cases} U = V \\ W = V^{-1}V' \end{cases} \tag{15}$$

We suppose that components in V, U, W are expressed as in section 2. But we express the components in V' as

$$V' = \begin{bmatrix} z'_{0,0} & \cdots & z'_{0,n-r-1} \\ \vdots & & \vdots \\ z'_{r-1,0} & \cdots & z'_{r-1,n-r-1} \end{bmatrix}.$$

Then (15) means that

$$\begin{cases} z_{i,j} = u_{i,j} \\ z'_{i,p} = \sum_j u_{i,j} w_{j,p} \end{cases} \qquad \begin{cases} u_{i,j} = z_{i,j} \\ w_{i,p} = \sum_j \bar{z}_{i,j} z'_{j,p} \end{cases} \tag{16}$$

$$(0 \leq i, j < r, \quad 0 \leq p < n - r).$$

where $\bar{z}_{i,j} = h\Delta_{j,i}$ because $V^{-1} = [\bar{z}_{i,j}] = (detU)^{-1}[\Delta_{i,j}]^t = h[\Delta_{j,i}]$.

For the coordinate change (16) between Z_λ and $GL(r) \times U_\lambda$, we obtain the next Lemma.

Lemma 6. *Let $F(Z) \in S_A$. By the relation $F(Z) = F(U[I\ W]) = h(U)\hat{F}(W)$, we obtain the function $h(U)\hat{F}(W)$ on $GL(r) \times U_\lambda$. Differentiations of $F(Z)$ and differentiations of $h(U)\hat{F}(W)$ are related as follows:*

(1) $$\left(\sum_{i=0}^{r-1} z_{i,k} \frac{\partial}{\partial z_{i,j}}\right) F = -\delta_{k,j} h\hat{F} - h \sum_{p=0}^{n-r-1} w_{j,p} \frac{\partial \hat{F}}{\partial w_{k,p}} \qquad (0 \leq k, j < r)$$

(2) $$\left(\sum_{i=0}^{r-1} z_{i,s} \frac{\partial}{\partial z'_{i,q}}\right) F = h \frac{\partial \hat{F}}{\partial w_{s,q}} \qquad (0 \leq s < r,\ 0 \leq q < n - r)$$

(3) $$\left(\sum_{i=0}^{r-1} z'_{i,s} \frac{\partial}{\partial z'_{i,q}}\right) F = h \sum_{i=0}^{r-1} w_{i,s} \frac{\partial \hat{F}}{\partial w_{i,q}} \qquad (0 \leq s, q < n - r)$$

(4) $$\frac{\partial}{\partial w_{j,p}} = \sum_{i=0}^{r-1} u_{i,j} \frac{\partial}{\partial z'_{i,p}} \qquad (0 \leq j < r,\ 0 \leq p < n - r)$$

(5) $$\frac{\partial}{\partial u_{i,j}} = \frac{\partial}{\partial z_{i,j}} + \sum_{p=0}^{n-r-1} w_{j,p} \frac{\partial}{\partial z'_{i,p}} \qquad (0 \leq i, j < r)$$

(6) $\dfrac{\partial F}{\partial z'_{i,q}} = h^2 \displaystyle\sum_{\bar{i}=0}^{r-1} \Delta_{i,\bar{i}} \dfrac{\partial \hat{F}}{\partial w_{\bar{i},q}} \qquad (0 \leq i < r,\ 0 \leq q < n-r)$

Proof. (1) From (16), Lemma 3 (2) and Lemma 5 (1), we have

$$\begin{aligned}
\frac{\partial}{\partial z_{i,j}} &= \frac{\partial}{\partial u_{i,j}} + \sum_{\bar{i},p} \frac{\partial w_{\bar{i},p}}{\partial z_{i,j}} \frac{\partial}{\partial w_{\bar{i},p}} \\
&= \frac{\partial}{\partial u_{i,j}} + \sum_{\bar{i},p} \left(\sum_{\bar{j}} \frac{\partial h \Delta_{\bar{j},\bar{i}}}{\partial z_{i,j}} z'_{\bar{j},p} \right) \frac{\partial}{\partial w_{\bar{i},p}} \\
&= \frac{\partial}{\partial u_{i,j}} + \sum_{\bar{i},p} \left[\sum_{\bar{j}} \left\{ (-h^2 \Delta_{i,j}) \Delta_{\bar{j},\bar{i}} + h \Delta_{\bar{j},\bar{i},i,j} \right\} z'_{\bar{j},p} \right] \frac{\partial}{\partial w_{\bar{i},p}} \\
&= \frac{\partial}{\partial u_{i,j}} + (-h\Delta_{i,j}) \sum_{\bar{i},p} \left(\sum_{\bar{j}} \bar{z}_{\bar{i},\bar{j}} z'_{\bar{j},p} \right) \frac{\partial}{\partial w_{\bar{i},p}} + h \sum_{\bar{i},p,\bar{j}} \Delta_{\bar{j},\bar{i},i,j} z'_{\bar{j},p} \frac{\partial}{\partial w_{\bar{i},p}} \\
&= \frac{\partial}{\partial u_{i,j}} + (-h\Delta_{i,j}) \sum_{\bar{i},p} w_{\bar{i},p} \frac{\partial}{\partial w_{\bar{i},p}} + h \sum_{\bar{i},p,\bar{j}} \Delta_{\bar{j},\bar{i},i,j} z'_{\bar{j},p} \frac{\partial}{\partial w_{\bar{i},p}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_i z_{i,k} \frac{\partial}{\partial z_{i,j}} \\
&= \sum_i u_{i,k} \frac{\partial}{\partial u_{i,j}} + (-h) \left(\sum_i \Delta_{i,j} u_{i,k} \right) \sum_{\bar{i},p} w_{\bar{i},p} \frac{\partial}{\partial w_{\bar{i},p}} + h \sum_{\bar{i},p,\bar{j},i} \Delta_{\bar{j},\bar{i},i,j} u_{i,k} z'_{\bar{j},p} \frac{\partial}{\partial w_{\bar{i},p}} \\
&= \sum_i u_{i,k} \frac{\partial}{\partial u_{i,j}} + (-\delta_{j,k}) \sum_{\bar{i},p} w_{\bar{i},p} \frac{\partial}{\partial w_{\bar{i},p}} + h \sum_{\bar{i},p,\bar{j},i} \Delta_{\bar{j},\bar{i},i,j} u_{i,k} z'_{\bar{j},p} \frac{\partial}{\partial w_{\bar{i},p}}.
\end{aligned}$$

Here we consider two cases.

(i) The case $k = j$.

Since

$$\begin{aligned}
&h \sum_{\bar{i},p,\bar{j},i} \Delta_{\bar{j},\bar{i},i,j} u_{i,j} z'_{\bar{j},p} \frac{\partial}{\partial w_{\bar{i},p}} \\
&= h \sum_{p,\bar{j}} \sum_{\bar{i} \neq j} \Delta_{\bar{j},\bar{i}} z'_{\bar{j},p} \frac{\partial}{\partial w_{\bar{i},p}} = \sum_{p,\bar{j}} \sum_{\bar{i} \neq j} \bar{z}_{\bar{i},\bar{j}} z'_{\bar{j},p} \frac{\partial}{\partial w_{\bar{i},p}} = \sum_{p} \sum_{\bar{i} \neq j} w_{\bar{i},p} \frac{\partial}{\partial w_{\bar{i},p}},
\end{aligned}$$

we have

$$\sum_i z_{i,j} \frac{\partial}{\partial z_{i,j}} = \sum_i u_{i,j} \frac{\partial}{\partial u_{i,j}} - \sum_{\bar{i},p} w_{\bar{i},p} \frac{\partial}{\partial w_{\bar{i},p}} + \sum_{\bar{i},p,\bar{i}\neq j} w_{\bar{i},p} \frac{\partial}{\partial w_{\bar{i},p}}$$

$$\sum_i z_{i,j} \frac{\partial}{\partial z_{i,j}} = \sum_i u_{i,j} \frac{\partial}{\partial u_{i,j}} - \sum_p w_{j,p} \frac{\partial}{\partial w_{j,p}}.$$

Then, by Lemma 3 (2), it holds that

$$\begin{aligned}\left(\sum_i z_{i,j}\frac{\partial}{\partial z_{i,j}}\right)F \quad &= \hat{F}\sum_i u_{i,j}\frac{\partial h}{\partial u_{i,j}} - h\sum_p w_{j,p}\frac{\partial \hat{F}}{\partial w_{j,p}}\\ &= (-\hat{F}h^2)(detU) - h\sum_p w_{j,p}\frac{\partial \hat{F}}{\partial w_{j,p}}.\end{aligned}$$

Therefore, we obtain

$$\left(\sum_i z_{i,j}\frac{\partial}{\partial z_{i,j}}\right)F = -h\hat{F} - h\sum_p w_{j,p}\frac{\partial \hat{F}}{\partial w_{j,p}}. \tag{17}$$

(ii) The case $k \neq j$.

In this case, it holds that

$$\sum_i z_{i,k}\frac{\partial}{\partial z_{i,j}} = \sum_i u_{i,k}\frac{\partial}{\partial u_{i,j}} + h\sum_{\bar{i},p,\bar{j},i}\Delta_{\bar{j},\bar{i},i,j}u_{i,k}z'_{\bar{j},p}\frac{\partial}{\partial w_{\bar{i},p}}.$$

From $k \neq j$, we have

$$\sum_i \Delta_{\bar{j},\bar{i},i,j}u_{i,k} = \begin{cases} 0 & (\text{if } \bar{i} \neq k)\\ (-1)^{j+k+|k-j|-1}\Delta_{\bar{j},j} = -\Delta_{\bar{j},j} & (\text{if } \bar{i} = k),\end{cases}$$

and

$$\begin{aligned}&h\sum_{\bar{i},p,\bar{j},i}\Delta_{\bar{j},\bar{i},i,j}u_{i,k}z'_{\bar{j},p}\frac{\partial}{\partial w_{\bar{i},p}}\\ &= h\sum_{p,\bar{j}}z'_{\bar{j},p}\left[\sum_{\bar{i}}\left(\sum_i \Delta_{\bar{j},\bar{i},i,j}u_{i,k}\right)\frac{\partial}{\partial w_{\bar{i},p}}\right] = h\sum_{p,\bar{j}}z'_{\bar{j},p}\left(\sum_i \Delta_{\bar{j},k,i,j}u_{i,k}\right)\frac{\partial}{\partial w_{k,p}}\\ &= h\sum_{p,\bar{j}}z'_{\bar{j},p}(-1)\Delta_{\bar{j},j}\frac{\partial}{\partial w_{k,p}} = (-1)\sum_p w_{j,p}\frac{\partial}{\partial w_{k,p}}.\end{aligned}$$

So, we have

$$\sum_i z_{i,k}\frac{\partial}{\partial z_{i,j}} = \sum_i u_{i,k}\frac{\partial}{\partial u_{i,j}} - \sum_p w_{j,p}\frac{\partial}{\partial w_{k,p}} \quad (k \neq j).$$

Then, it holds that

$$\begin{aligned}\left(\sum_i z_{i,k}\frac{\partial}{\partial z_{i,j}}\right)F \quad &= \hat{F}\sum_i u_{i,k}\frac{\partial h}{\partial u_{i,j}} - h\sum_p w_{j,p}\frac{\partial \hat{F}}{\partial w_{k,p}}\\ &= \hat{F}\sum_i u_{i,k}(-h^2\Delta_{i,j}) - h\sum_p w_{j,p}\frac{\partial \hat{F}}{\partial w_{k,p}}.\end{aligned}$$

Because $k \neq j$, we obtain

$$\left(\sum_i z_{i,k}\frac{\partial}{\partial z_{i,j}}\right)F = -h\sum_p w_{j,p}\frac{\partial \hat{F}}{\partial w_{k,p}}. \tag{18}$$

From (17),(18), the following desired result is obtained :

$$\left(\sum_i z_{i,k}\frac{\partial}{\partial z_{i,j}}\right)F = -\delta_{k,j}h\hat{F} - h\sum_p w_{j,p}\frac{\partial \hat{F}}{\partial w_{k,p}}.$$

(2) First, from

$$\frac{\partial}{\partial z'_{i,q}} = \sum_{\bar{i},p}\frac{\partial w_{\bar{i},p}}{\partial z'_{i,q}}\frac{\partial}{\partial w_{\bar{i},p}} = \sum_{\bar{i},p}\left(\sum_{\bar{j}}\bar{z}_{\bar{i},\bar{j}}\frac{\partial z'_{\bar{j},p}}{\partial z'_{i,q}}\right)\frac{\partial}{\partial w_{\bar{i},p}},$$

we obtain

$$\frac{\partial}{\partial z'_{i,q}} = \sum_{\bar{i}}\bar{z}_{\bar{i},i}\frac{\partial}{\partial w_{\bar{i},q}}. \tag{19}$$

Using this equation, we get

$$\begin{aligned}\sum_i z_{i,s}\frac{\partial}{\partial z'_{i,q}} &= \sum_{i,\bar{i}} z_{i,s}\bar{z}_{\bar{i},i}\frac{\partial}{\partial w_{\bar{i},q}} = \sum_{i,\bar{i}} u_{i,s}h\Delta_{i,\bar{i}}\frac{\partial}{\partial w_{\bar{i},q}}\\ &= h\sum_{\bar{i}}\delta_{s,\bar{i}}(detU)\frac{\partial}{\partial w_{\bar{i},q}} = \frac{\partial}{\partial w_{s,q}}.\end{aligned}$$

Then, it holds that

$$\left(\sum_i z_{i,s}\frac{\partial}{\partial z'_{i,q}}\right)F = h\frac{\partial \hat{F}}{\partial w_{s,q}}.$$

(3) From equation (19), we can deduce

$$\sum_i z'_{i,s}\frac{\partial}{\partial z'_{i,q}} = \sum_{\bar{i}}\left(\sum_i \bar{z}_{\bar{i},i}z'_{i,s}\right)\frac{\partial}{\partial w_{\bar{i},q}} = \sum_{\bar{i}} w_{\bar{i},s}\frac{\partial}{\partial w_{\bar{i},q}}.$$

Then, we have

$$\left(\sum_i z'_{i,s}\frac{\partial}{\partial z'_{i,q}}\right)F = h\sum_i w_{i,s}\frac{\partial \hat{F}}{\partial w_{i,q}}.$$

(4) The equation

$$\begin{aligned}\frac{\partial}{\partial w_{j,p}} &= \sum_{i,q}\frac{\partial z'_{i,q}}{\partial w_{j,p}}\frac{\partial}{\partial z'_{i,q}} = \sum_{i,q}\left(\sum_{\bar{j}} u_{i,\bar{j}}\frac{\partial w_{\bar{j},q}}{\partial w_{j,p}}\right)\frac{\partial}{\partial z'_{i,q}}\\ &= \sum_{i,q}\sum_{\bar{j}} u_{i,\bar{j}}\delta_{\bar{j},j}\delta_{q,p}\frac{\partial}{\partial z'_{i,q}}\end{aligned}$$

implies

$$\frac{\partial}{\partial w_{j,p}} = \sum_i u_{i,j}\frac{\partial}{\partial z'_{i,p}}.$$

(5) The equation

$$\frac{\partial}{\partial u_{i,j}} = \frac{\partial}{\partial z_{i,j}} + \sum_{\bar{i},p}\frac{\partial z'_{\bar{i},p}}{\partial u_{i,j}}\frac{\partial}{\partial z'_{\bar{i},p}} = \frac{\partial}{\partial z_{i,j}} + \sum_{\bar{i},p}\left(\sum_{\bar{j}}\frac{\partial u_{\bar{i},\bar{j}}}{\partial u_{i,j}}w_{\bar{j},p}\right)\frac{\partial}{\partial z'_{\bar{i},p}}$$

implies

$$\frac{\partial}{\partial u_{i,j}} = \frac{\partial}{\partial z_{i,j}} + \sum_p w_{j,p} \frac{\partial}{\partial z'_{i,p}}.$$

(6) From (19), it holds that

$$\frac{\partial F}{\partial z'_{i,q}} = h \sum_{\bar{i}} \bar{z}_{\bar{i},i} \frac{\partial \hat{F}}{\partial w_{\bar{i},q}} = h^2 \sum_{\bar{i}} \Delta_{i,\bar{i}} \frac{\partial \hat{F}}{\partial w_{\bar{i},q}}.$$

□

3.5 Proof of Theorem 1

From now on, we will prove Theorem 1. First, we derive $C_{\lambda,\alpha}$ from $G_{\lambda,\alpha}$. Next, we derive $G_{\lambda,\alpha}$ from $C_{\lambda,\alpha}$.

Proof of Theorem 1. [From $G_{\lambda,\alpha}$ to $C_{\lambda,\alpha}$]

Let $F(Z) \in S$. Then by Proposition B, $F(Z) \in S \subset S_{A,B} \subset S_A$. So $F(Z)$ is written as $F(Z) = F(U[I\ W]) = h(U)\hat{F}(W)$. We will show $\hat{F}(W)$ satisfies $C_{\lambda,\alpha}$. We derive equations (1),(2), $\cdots$, (5) in $C_{\lambda,\alpha}$ from $G_{\lambda,\alpha}$ in this order.

(1) If $r \leq A_k$, $Z = [V\ V']$ is expressed as

$$\begin{aligned} Z &= \left[\overbrace{\begin{array}{ll|} z_{0,0} & \cdots \\ \vdots & \\ z_{r-1,0} & \cdots \end{array}}^{V} \; \overbrace{\begin{array}{c|lllll|c} \cdots & z_{0,A_k} & \cdots & z_{0,A_k+m} & \cdots & z_{0,A_{k+1}-1} & \cdots \\ & \vdots & & \vdots & & \vdots & \\ & z_{r-1,A_k} & \cdots & z_{r-1,A_k+m} & \cdots & z_{r-1,A_{k+1}-1} & \end{array}}^{V'} \right] \\ &= U[I\ W] \\ &= U \left[\overbrace{\begin{array}{ccc} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{array}}^{I} \middle| \overbrace{\begin{array}{lllll} w_{0,0} & \cdots\cdots & w_{0,q} & \cdots\cdots & w_{0,n-r-1} \\ \vdots & & \vdots & & \vdots \\ w_{r-1,0} & \cdots\cdots & w_{r-1,q} & \cdots\cdots & w_{r-1,n-r-1} \end{array}}^{W} \right] \end{aligned}$$

In this case, $L_{k,m}F$ is expressed as

$$L_{k,m}F = \sum_i \sum_{p=A_k+m}^{A_{k+1}-1} z'_{i,p-r-m} \frac{\partial F}{\partial z'_{i,p-r}},$$

because $z_{i,p} = z'_{i,p-r}$ $(p \geq r)$.

By Lemma 6 (3), we obtain

$$L_{k,m}F = \sum_{p=A_k+m}^{A_{k+1}-1} \left(\sum_i z'_{i,p-r-m} \frac{\partial F}{\partial z'_{i,p-r}} \right) = h \sum_{p=A_k+m}^{A_{k+1}-1} \left(\sum_i w_{i,p-r-m} \frac{\partial \hat{F}}{\partial w_{i,p-r}} \right).$$

Then, it holds that

$$\begin{aligned} \alpha_m^{(k)} h\hat{F} &= h\hat{L}_{k,m}\hat{F} \\ \hat{L}_{k,m}\hat{F} &= \alpha_m^{(k)}\hat{F} \quad (r \leq A_k). \end{aligned}$$

Thus $C_{\lambda,\alpha}$ (1) has been deduced.

(2) Suppose $A_k < r \leq A_k + m$. In this case, $L_{k,m}$ is expressed as

$$\begin{aligned}
L_{k,m}F &= \sum_i \sum_{p=A_k+m}^{r+m-1} z_{i,p-m} \frac{\partial F}{\partial z_{i,p}} + \sum_i \sum_{p=r+m}^{A_{k+1}-1} z_{i,p-m} \frac{\partial F}{\partial z_{i,p}} \\
&= \sum_i \sum_{p=A_k+m}^{r+m-1} z_{i,p-m} \frac{\partial F}{\partial z'_{i,p-r}} + \sum_i \sum_{p=r+m}^{A_{k+1}-1} z'_{i,p-m-r} \frac{\partial F}{\partial z'_{i,p-r}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{if } r+m < A_{k+1}) \\
\text{or} & \\
L_{k,m}F &= \sum_i \sum_{p=A_k+m}^{A_{k+1}-1} z_{i,p-m} \frac{\partial F}{\partial z'_{i,p-r}} \qquad\qquad (\text{if } r+m \geq A_{k+1}).
\end{aligned}$$

We will rewrite each part of the above equations into a equation of h and $\hat{F}$. By Lemma 6 (2),(3), we obtain

$$\sum_{p=A_k+m}^{r+m-1} \left(\sum_i z_{i,p-m} \frac{\partial F}{\partial z'_{i,p-r}} \right) = h \sum_{p=A_k+m}^{r+m-1} \frac{\partial \hat{F}}{\partial w_{p-m,p-r}}.$$

$$\sum_{p=r+m}^{A_{k+1}-1} \left(\sum_i z'_{i,p-m-r} \frac{\partial F}{\partial z'_{i,p-r}} \right) = h \sum_{p=r+m}^{A_{k+1}-1} \sum_i w_{i,p-m-r} \frac{\partial \hat{F}}{\partial w_{i,p-r}}.$$

Then, if $r + m < A_{k+1}$, we have

$$L_{k,m}F = h \left[\sum_{p=A_k+m}^{r+m-1} \frac{\partial}{\partial w_{p-m,p-r}} + \sum_{p=r+m}^{A_{k+1}-1} \sum_i w_{i,p-m-r} \frac{\partial}{\partial w_{i,p-r}} \right] \hat{F}.$$

Similarly, if $r + m \geq A_{k+1}$, we have

$$L_{k,m}F = h \left[\sum_{p=A_k+m}^{A_{k+1}-1} \frac{\partial}{\partial w_{p-m,p-r}} \right] \hat{F}.$$

From these equations, it holds that

$$\alpha_m^{(k)} h \hat{F} = h I_{k,m} \hat{F}.$$

Therefore, when $A_k < r \leq A_k + m$, equation

$$I_{k,m} \hat{F} = \alpha_m^{(k)} \hat{F}$$

holds.

(3) Suppose $A_k + m < r \leq A_{k+1} - 1$. In this case, $L_{k,m}$ is expressed as

$$\begin{aligned} L_{k,m}F &= \sum_i \sum_{p=A_k+m}^{r-1} z_{i,p-m} \frac{\partial F}{\partial z_{i,p}} + \sum_i \sum_{p=r}^{r+m-1} z_{i,p-m} \frac{\partial F}{\partial z'_{i,p-r}} \\ &+ \sum_i \sum_{p=r+m}^{A_{k+1}-1} z'_{i,p-m-r} \frac{\partial F}{\partial z'_{i,p-r}} \qquad (\text{if } r+m < A_{k+1}) \end{aligned}$$

or

$$\begin{aligned} L_{k,m}F &= \sum_i \sum_{p=A_k+m}^{r-1} z_{i,p-m} \frac{\partial F}{\partial z_{i,p}} + \sum_i \sum_{p=r}^{A_{k+1}-1} z_{i,p-m} \frac{\partial F}{\partial z'_{i,p-r}} \\ &\qquad\qquad\qquad\qquad (\text{if } r+m \geq A_{k+1}). \end{aligned}$$

By Lemma 6 (1),(2) and(3), we obtain

$$\begin{aligned} \sum_{p=A_k+m}^{r-1} \left[\sum_{i=0}^{r-1} z_{i,p-m} \frac{\partial F}{\partial z_{i,p}} \right] &= \sum_{p=A_k+m}^{r-1} \left[-\delta_{p-m,p} h \hat{F} - h \sum_{q=0}^{n-r-1} w_{p,q} \frac{\partial \hat{F}}{\partial w_{p-m,q}} \right] \\ &= (-h) \left[\delta_{m,0}(r-A_k) + \sum_{p=A_k+m}^{r-1} \sum_{q=0}^{n-r-1} w_{p,q} \frac{\partial}{\partial w_{p-m,q}} \right] \hat{F} \end{aligned}$$

$$\sum_{p=r}^{r+m-1} \left[\sum_i z_{i,p-m} \frac{\partial F}{\partial z'_{i,p-r}} \right] = h \left[\sum_{p=r}^{r+m-1} \frac{\partial}{\partial w_{p-m,p-r}} \right] \hat{F}$$

$$\sum_{p=r+m}^{A_{k+1}-1} \left[\sum_i z'_{i,p-m-r} \frac{\partial F}{\partial z'_{i,p-r}} \right] = h \left[\sum_{p=r+m}^{A_{k+1}-1} \sum_i w_{i,p-m-r} \frac{\partial}{\partial w_{i,p-r}} \right] \hat{F}.$$

From these equations, if $r+m < A_{k+1}$, we obtain

$$\begin{aligned} L_{k,m}F &= h \left[-\delta_{m,0}(r-A_k) - \sum_{p=A_k+m}^{r-1} \sum_{q=0}^{n-r-1} w_{p,q} \frac{\partial}{\partial w_{p-m,q}} \right. \\ &\left. + \sum_{p=r}^{r+m-1} \frac{\partial}{\partial w_{p-m,p-r}} + \sum_{p=r+m}^{A_{k+1}-1} \sum_{i=0}^{r-1} w_{i,p-m-r} \frac{\partial}{\partial w_{i,p-r}} \right] \hat{F}. \end{aligned}$$

Similarly, if $r+m \geq A_{k+1}$, we obtain

$$\begin{aligned} L_{k,m}F &= h \left[-\delta_{m,0}(r-A_k) - \sum_{p=A_k+m}^{r-1} \sum_{q=0}^{n-r-1} w_{p,q} \frac{\partial}{\partial w_{p-m,q}} \right. \\ &\left. + \sum_{p=r}^{A_{k+1}-1} \frac{\partial}{\partial w_{p-m,p-r}} \right] \hat{F}. \end{aligned}$$

From these equations,

$$\alpha_m^{(k)} h \hat{F} = -h[\delta_{m,0}(r-A_k)] + h J_{k,m} \hat{F}$$

is deduced. Therefore it holds that

$$J_{k,m} \hat{F} = [\delta_{m,0}(r-A_k) + \alpha_m^{(k)}] \hat{F}.$$

(4) Suppose $A_{k+1} \leq r$. In this case, $L_{k,m}$ is written as

$$L_{k,m}F = \sum_{i} \sum_{p=A_k+m}^{A_{k+1}-1} z_{i,p-m} \frac{\partial F}{\partial z_{i,p}}.$$

By Lemma 6 (1), we obtain

$$\begin{aligned}
\sum_{p=A_k+m}^{A_{k+1}-1} \left(\sum_{i} z_{i,p-m} \frac{\partial F}{\partial z_{i,p}} \right) &= \sum_{p=A_k+m}^{A_{k+1}-1} \left[-\delta_{p-m,p} h\hat{F} - h \sum_{q=0}^{n-r-1} w_{p,q} \frac{\partial \hat{F}}{\partial w_{p-m,q}} \right] \\
&= (-h) \left[\delta_{m,0}(A_{k+1} - A_k) + \sum_{p=A_k+m}^{A_{k+1}-1} \sum_{q=0}^{n-r-1} w_{p,q} \frac{\partial}{\partial w_{p-m,q}} \right] \hat{F}.
\end{aligned}$$

Then, we have

$$L_{k,m}F = (-h) \left[\delta_{m,0}(A_{k+1} - A_k) + \sum_{p=A_k+m}^{A_{k+1}-1} \sum_{q=0}^{n-r-1} w_{p,q} \frac{\partial}{\partial w_{p-m,q}} \right] \hat{F}$$

$$\alpha_m^{(k)} h\hat{F} = (-h) \left[\delta_{m,0}\lambda_k + K_{k,m} \right] \hat{F}.$$

Therefore, equation

$$K_{k,m}\hat{F} = -[\delta_{m,0}\lambda_k + \alpha_m^{(k)}]\hat{F}$$

is deduced.

(5) Here we derive $C_{\lambda,\alpha}(5)$. From Lemma 6 (4), we obtain

$$\frac{\partial F}{\partial w_{j,q}} = \sum_{\bar{j}} u_{\bar{j},j} \frac{\partial F}{\partial z'_{\bar{j},q}}$$

$$\frac{\partial}{\partial w_{i,p}} \left[h \frac{\partial \hat{F}}{\partial w_{j,q}} \right] = \sum_{\bar{i}} u_{\bar{i},i} \frac{\partial}{\partial z'_{\bar{i},p}} \left[\sum_{\bar{j}} u_{\bar{j},j} \frac{\partial F}{\partial z'_{\bar{j},q}} \right]$$

$$h \frac{\partial^2 \hat{F}}{\partial w_{i,p} \partial w_{j,q}} = \sum_{\bar{i}} \sum_{\bar{j}} u_{\bar{i},i} u_{\bar{j},j} \frac{\partial^2 F}{\partial z'_{\bar{i},p} \partial z'_{\bar{j},q}}.$$

Therefore, it holds that

$$\begin{aligned}
&h \left[\frac{\partial^2 \hat{F}}{\partial w_{i,p} \partial w_{j,q}} - \frac{\partial^2 \hat{F}}{\partial w_{i,q} \partial w_{j,p}} \right] \\
&= \sum_{\bar{i}} \sum_{\bar{j}} u_{\bar{i},i} u_{\bar{j},j} \frac{\partial^2 F}{\partial z'_{\bar{i},p} \partial z'_{\bar{j},q}} - \sum_{\bar{i}} \sum_{\bar{j}} u_{\bar{i},i} u_{\bar{j},j} \frac{\partial^2 F}{\partial z'_{\bar{i},q} \partial z'_{\bar{j},p}} \\
&= \sum_{\bar{i}} \sum_{\bar{j}} u_{\bar{i},i} u_{\bar{j},j} \left[\frac{\partial^2 F}{\partial z'_{\bar{i},p} \partial z'_{\bar{j},q}} - \frac{\partial^2 F}{\partial z'_{\bar{i},q} \partial z'_{\bar{j},p}} \right] = \sum_{\bar{i}} \sum_{\bar{j}} u_{\bar{i},i} u_{\bar{j},j} \Box_{\bar{i},\bar{j},p,q} F = 0.
\end{aligned}$$

So we obtain

$$\hat{\Box}_{i,j,p,q}\hat{F} = 0 \quad (0 \leq i, j \leq r-1,\ 0 \leq p, q \leq n-r-1).$$

[From $C_{\lambda,\alpha}$ to $G_{\lambda,\alpha}$]

Let $\hat{F}(W)$ be a solution of $C_{\lambda,\alpha}$. We define $F(Z)(\in S_A)$ as

$$F(Z) = F(U[I\ W]) = h(U)\hat{F}(W) \qquad (U \in GL(r)).$$

We will show $F(Z)$ satisfies $G_{\lambda,\alpha}$.

(a) When $r \leq A_k$, from the consideration in the above (1), we have $L_{k,m}F = h\hat{L}_{k,m}\hat{F}$. Since $\hat{L}_{k,m}\hat{F} = \alpha_m^{(k)}\hat{F}$, it holds that

$$L_{k,m}F = h\hat{L}_{k,m}\hat{F} = h\alpha_m^{(k)}\hat{F} = \alpha_m^{(k)}F.$$

By similar considerations, we can obtain $L_{k,m}F = \alpha_m^{(k)}F \ \ (0 \leq k \leq l-1,\ 0 \leq m \leq \lambda_k - 1)$ from $C_{\lambda,\alpha}(1), (2), (3), (4)$.

(b) By Proposition 4, it holds that $M_{i,j}F = -\delta_{i,j}F \ \ (0 \leq i, j \leq r-1)$.

(c) In order to derive $\Box_{i,j,p,q}F = 0$, we consider four cases.

(i) Suppose $p < r$ and $q < r$. $\Box_{i,j,p,q}F$ is written as

$$\begin{aligned} \Box_{i,j,p,q}F &= \left(\frac{\partial^2}{\partial u_{i,p}\partial u_{j,q}} - \frac{\partial^2}{\partial u_{i,q}\partial u_{j,p}}\right) h(U)\hat{F}(W) \\ &= \hat{F}(W)\left(\frac{\partial^2 h}{\partial u_{i,p}\partial u_{j,q}} - \frac{\partial^2 h}{\partial u_{i,q}\partial u_{j,p}}\right) = 0, \end{aligned}$$

because of Lemma 3 (1).

(ii) Suppose $p \geq r$ and $q \geq r$. Putting $\bar{p} = p - r, \bar{q} = q - r$, we have

$$\Box_{i,j,p,q}F = \left(\frac{\partial^2}{\partial z'_{i,\bar{p}}\partial z'_{j,\bar{q}}} - \frac{\partial^2}{\partial z'_{i,\bar{q}}\partial z'_{j,\bar{p}}}\right) F.$$

From Lemma 6 (6), we obtain

$$\frac{\partial F}{\partial z'_{j,\bar{q}}} = h^2 \sum_{\bar{i}} \Delta_{j,\bar{i}} \frac{\partial \hat{F}}{\partial w_{\bar{i},\bar{q}}}.$$

Then we have

$$\begin{aligned} \frac{\partial^2 F}{\partial z'_{i,\bar{p}}\partial z'_{j,\bar{q}}} &= \frac{\partial}{\partial z'_{i,\bar{p}}}\left[h^2 \sum_{\bar{i}} \Delta_{j,\bar{i}} \frac{\partial \hat{F}}{\partial w_{\bar{i},\bar{q}}}\right] = h^2 \sum_{\bar{i}} \Delta_{j,\bar{i}} \sum_{d,h} \frac{\partial^2 \hat{F}}{\partial w_{d,h}\partial w_{\bar{i},\bar{q}}} \sum_{\bar{j}} \bar{z}_{d,\bar{j}} \frac{\partial z'_{\bar{j},h}}{\partial z'_{i,\bar{p}}} \\ &= h^2 \sum_{\bar{i}} \Delta_{j,\bar{i}} \sum_{d} \frac{\partial^2 \hat{F}}{\partial w_{d,\bar{p}}\partial w_{\bar{i},\bar{q}}} \bar{z}_{d,i} = h^3 \sum_{\bar{i},d} \Delta_{j,\bar{i}}\Delta_{i,d} \frac{\partial^2 \hat{F}}{\partial w_{d,\bar{p}}\partial w_{\bar{i},\bar{q}}}. \end{aligned}$$

Similarly, we have

$$\frac{\partial^2 F}{\partial z'_{i,\bar{q}}\partial z'_{j,\bar{p}}} = h^3 \sum_{\bar{i},d} \Delta_{j,\bar{i}}\Delta_{i,d} \frac{\partial^2 \hat{F}}{\partial w_{d,\bar{q}}\partial w_{\bar{i},\bar{p}}}.$$

Then we obtain

$$\begin{aligned}\Box_{i,j,p,q}F &= h^3\sum_{\bar{i},d}\Delta_{j,\bar{i}}\Delta_{i,d}\left[\frac{\partial^2\hat{F}}{\partial w_{d,\bar{p}}\partial w_{\bar{i},\bar{q}}}-\frac{\partial^2\hat{F}}{\partial w_{d,\bar{q}}\partial w_{\bar{i},\bar{p}}}\right]\\ &= h^3\sum_{\bar{i},d}\Delta_{j,\bar{i}}\Delta_{i,d}\hat{\Box}_{d,\bar{i},\bar{p},\bar{q}}\hat{F}=0.\end{aligned}$$

(iii) Suppose $p < r$ and $q \geq r$. In this case, putting $\bar{q} = q - r$, $\Box_{i,j,p,q}F$ is written as

$$\Box_{i,j,p,q}F=\left(\frac{\partial^2}{\partial u_{i,p}\partial z'_{j,\bar{q}}}-\frac{\partial^2}{\partial z'_{i,\bar{q}}\partial u_{j,p}}\right)F.$$

From Lemma 6 (6),

$$\frac{\partial F}{\partial z'_{j,\bar{q}}}=h^2\sum_{\bar{i}}\Delta_{j,\bar{i}}\frac{\partial\hat{F}}{\partial w_{\bar{i},\bar{q}}}.$$

Then we have

$$\begin{aligned}\frac{\partial^2 F}{\partial u_{i,p}\partial z'_{j,\bar{q}}} &= \frac{\partial}{\partial u_{i,p}}\left[h^2\sum_{\bar{i}}\Delta_{j,\bar{i}}\frac{\partial\hat{F}}{\partial w_{\bar{i},\bar{q}}}\right]\\ &= 2h[-h^2\Delta_{i,p}]\sum_{\bar{i}}\Delta_{j,\bar{i}}\frac{\partial\hat{F}}{\partial w_{\bar{i},\bar{q}}}+h^2\sum_{\bar{i}}\Delta_{j,\bar{i},i,p}\frac{\partial\hat{F}}{\partial w_{\bar{i},\bar{q}}}\\ &= h^2\sum_{\bar{i}}\left[(-2h)\Delta_{i,p}\Delta_{j,\bar{i}}+\Delta_{j,\bar{i},i,p}\right]\frac{\partial\hat{F}}{\partial w_{\bar{i},\bar{q}}}.\end{aligned}$$

Here we have used Lemma 3 (2) and Lemma 5 (1). Similarly, we have

$$\frac{\partial^2 F}{\partial z'_{i,\bar{q}}\partial u_{j,p}}=\frac{\partial^2 F}{\partial u_{j,p}\partial z'_{i,\bar{q}}}=h^2\sum_{\bar{i}}\left[(-2h)\Delta_{j,p}\Delta_{i,\bar{i}}+\Delta_{i,\bar{i},j,p}\right]\frac{\partial\hat{F}}{\partial w_{\bar{i},\bar{q}}}.$$

Therefore, it holds that

$$\begin{aligned}\Box_{i,j,p,q}F &= h^2\sum_{\bar{i}}\left[(-2h)[\Delta_{i,p}\Delta_{j,\bar{i}}-\Delta_{i,\bar{i}}\Delta_{j,p}]+\Delta_{j,\bar{i},i,p}-\Delta_{i,\bar{i},j,p}\right]\frac{\partial\hat{F}}{\partial w_{\bar{i},\bar{q}}}\\ &= h^2\sum_{\bar{i}}\left[(-2h)detU\Delta_{i,p,j,\bar{i}}+\Delta_{j,\bar{i},i,p}-\Delta_{i,\bar{i},j,p}\right]\frac{\partial\hat{F}}{\partial w_{\bar{i},\bar{q}}}\\ &= h^2\sum_{\bar{i}}\left[-2\Delta_{i,p,j,\bar{i}}+\Delta_{j,\bar{i},i,p}+\Delta_{i,p,j,\bar{i}}\right]\frac{\partial\hat{F}}{\partial w_{\bar{i},\bar{q}}}\\ &= h^2\sum_{\bar{i}}\left[-2\Delta_{i,p,j,\bar{i}}+2\Delta_{i,p,j,\bar{i}}\right]\frac{\partial\hat{F}}{\partial w_{\bar{i},\bar{q}}}=0.\end{aligned}$$

Here we have used Lemma 5 (2),(3),(4).

(iv) Suppose $p \geq r$ and $q < r$. Putting $\bar{p} = p - r$, $\Box_{i,j,p,q}F$ is written as

$$\begin{aligned}\Box_{i,j,p,q}F &= \left(\frac{\partial^2}{\partial z'_{i,\bar{p}}\partial u_{j,q}}-\frac{\partial^2}{\partial u_{i,q}\partial z'_{j,\bar{p}}}\right)F\\ &= -\left(\frac{\partial^2}{\partial u_{i,q}\partial z'_{j,\bar{p}}}-\frac{\partial^2}{\partial z'_{i,\bar{p}}\partial u_{j,q}}\right)F=0.\end{aligned}$$

The identity of the second line is obtained from the case (iii). □

3.6 Proof of Theorem 2

Proof of Theorem 2. Let $F(Z) \in S$. Then $F(Z)$ satisfies the equation

$$\Box_{ijpq}F(Z) = 0 \qquad (0 \leq i, j \leq r-1,\ 0 \leq p, q \leq n-1).$$

By Proposition 2 (2), it holds that

$$(\zeta_\lambda)_*(\Box_{ijpq})F(\zeta_\lambda^{-1}(UA, \mathbf{t}, B)) = 0 \\ (0 \leq i, j \leq r-1,\ 0 \leq p, q \leq n-1),$$

where $Z = UA[I\ T]B$. As

$$F(\zeta_\lambda^{-1}(UA, \mathbf{t}, B)) = F(UA[I\ T]B) = h(UA)f(\mathbf{t})\chi_\lambda(B, \alpha),$$

we obtain

$$(\zeta_\lambda)_*(\Box_{ijpq})h(UA)f(\mathbf{t})\chi_\lambda(B, \alpha) = 0 \\ (0 \leq i, j \leq r-1,\ 0 \leq p, q \leq n-1).$$

Therefore $h(UA)f(\mathbf{t})\chi_\lambda(B, \alpha)$ satisfies $\widetilde{H}_{\lambda,\alpha}$.

Conversely, suppose $f(\mathbf{t}) \in \mathcal{O}(D_\lambda)$ and that $h(UA)f(\mathbf{t})\chi_\lambda(B, \alpha)$ satisfies $\widetilde{H}_{\lambda,\alpha}$. We define an analytic function $F(Z)(\in \mathcal{O}(Z_\lambda))$ by $F(Z) = F(UA[I\ T]B) = h(UA)f(\mathbf{t})\chi_\lambda(B, \alpha)$.

From

$$(\zeta_\lambda)_*(\Box_{ijpq})h(UA)f(\mathbf{t})\chi_\lambda(B, \alpha) = (\zeta_\lambda)_*(\Box_{ijpq})F(\zeta_\lambda^{-1}(UA, \mathbf{t}, B)) = 0,$$

we obtain

$$(\zeta_\lambda^{-1})_*(\zeta_\lambda)_*(\Box_{ijpq})F(\zeta_\lambda^{-1}(\zeta_\lambda(Z))) = 0.$$

Then we have

$$\Box_{ijpq}F(Z) = 0 \quad (0 \leq i, j \leq r-1,\ 0 \leq p, q \leq n-1).$$

On the other hand, $F(Z) \in S_{A,B}$ by its definition and Proposition 3 (2). Then from Proposition 5, $F(Z)$ satisfies the condition

$$\begin{array}{ll} L_{km}F(Z) = \alpha_m^{(k)}F(Z) & (0 \leq k \leq l-1,\ 0 \leq m \leq \lambda_k - 1) \\ M_{ij}F(Z) = -\delta_{ij}F(Z) & (0 \leq i, j \leq r-1). \end{array}$$

Therefore, $F(Z) \in S$. □

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